Voting with Restricted Communication

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Nomenclature

$<$	total order on $\mathcal{K}$, $\mathcal{T}$ or $\mathbb{R}$
$\mathbb{1}$	all-one vector
α_i	phantom ballot
$\mathcal{F}$	range $g_{\text{second-best}}$
$\langle \bullet, \bullet \rangle$	Euclidean scalar product
$\mathcal{K}$	choice set
$\mathcal{M}$	message set
$\mathbf{t}$	type vector
med	Median
Ω	Landau symbol: Not faster asymptotically
$\bar{t}$	average type
$\mathcal{T}$	type set containing preferences
$\mathcal{T}_A$	preferences restricted to $A \subseteq \mathcal{K}$
$\tau_{\mathcal{K}}(t)$	top alternative on $\mathcal{K}$ of type t
$\underline{k}, \overline{k}$	upper resp. lower bound of $\mathcal{K}$
f	social choice function
$f^{\downarrow}$	signal to quantisation level
$f^{\uparrow}$	quantisation level to representative point
g	mechanism $\mathcal{M}^n \to \mathcal{K}$
k	$= \lvert M \rvert$, number of signals
m_i	message of agent i
n	number of agents
O	Landau symbol: at least as fast
p	projection $\mathcal{T} \twoheadrightarrow \mathcal{T}_A$
s_i	strategy of agent i $\mathcal{T} \to \mathcal{M}$
t_i	player i's type

1 Introduction

This thesis considers a model of voting where agents are restricted in their ability to communicate to the principal. Observing their private type, the agents choose one of a (small) finite number of signals that they send to the principal. Such communication restrictions are common in democratic electoral systems as well as in allocation problems on the internet. Common to all is that the type space of agents (be it voters or internet servers) is much more complex than what they can or want to communicate to a mechanism. Communication restrictions arise for different reasons.
First, it is costly to communicate the exact position in a large type space. In computing environments, economic decisions involve low-cost computational resources. The communication of even a fixed-length integer (this is a number in $0, 1, 2, \ldots, 2^{32} - 1$) might be too costly. A small number of different signals that agents communicate might hence be a desideratum and sometimes even a constraint for mechanism design for the internet.
Furthermore, agents might be hesitant to reveal their type. For example, in income and wealth questionnaires, depending on culture, direct questions for the income of an agent are omitted as these would not be answered—instead, the agents shall announce one of a certain number of intervals in which their income lies.
As a third example, there might be intellectual limits to the agents' perception. If there are 10,000 options, an agent might be overwhelmed by the number of alternatives and in fact only evaluate a few of the alternatives, e.g. the first and last few of them. In addition, informed debate of alternatives could often result in too high opportunity costs. Therefore, studying a model with restricted communication might yield valuable insights into real-world voting mechanisms.

Nevertheless, much of the mechanism design literature is based on the revelation principle. The revelation principle says that any mechanism can be implemented by the agents revealing their true type. The argument is that for any mechanism with a different communication structure, a "black box" could be introduced that takes the types of the agents and "plays" for them in an optimal way—assuming agents announce their type truthfully. For the reasons given above, the communication of one's type to such a black box is not feasible in many environments. Hence, one cannot consider direct revelation mechanisms in a realistic model of mechanism design with restricted communication.

A body of literature studies mechanism design with limited communication and with monetary transfers. In many situations, however, monetary transfers are not feasible: First, transaction costs might be too high to justify a monetary transfer. This might be the case if the economic decision to be taken is one of low stakes, e.g. if the agents decide on low-value computing resources. In other situations, monetary transactions are not used for ethical reasons, as in democratic elections. The present study contributes to the literature on mechanism design *without* monetary transfers.

Furthermore, the literature on voting can be divided into two branches. On the one hand, Bayesian Incentive Compatible (BIC) mechanism design studies implementability in Bayesian equilibria. As voting studies implementation in simultaneous-move games, the Bayesian approach assumes that the agents have sufficient prior information about the type distribution of the other agents. On the other hand, Dominant Incentive Compatible (DIC) mechanism design studies implementability in dominant strategies. The latter is more robust to different beliefs the agents might have about the other agent's types. In fact, a mechanism is DIC if and only if it is BIC with respect to any system of beliefs the agents might have. We ask whether incentive compatible, in particular DIC, mechanisms in a model with restricted communication can be complex enough to yield high welfare. An easy measure of complexity is the number of values a voting mechanism attains.

Separate from the above, this thesis also seeks to understand from a perspective of communication restrictions in which sense representation in democratic systems might be implied by incentive compatibility. Representation by a party means that each agent decides which of a finite set of parties should represent her. Votes are then taken by the parties. We ask whether such a representation has incentive-compatible alternatives.
In the history of election systems, it has been observed that different margins of victory lead to different policy choices by the winning party. This study of *mandates* hints in a direction that weak commitment might allow for more cmplex mechanisms. We will not study situations of weaker commitment, but leave this for further work.

The present thesis is to the best of our knowledge the first contribution to DIC mechanism design with restricted communication and without monetary transfers. Our contributions are threefold:

(a) In a one-dimensional voting model with quadratic utilities, we give a strong necessary condition that first-best mechanisms under restricted communication have to fulfil. We provide upper and lower bounds on the rate of convergence of average welfare in the size of the message set and show that the range of any first-best mechanism grows with the size of the society.

(b) For the single-peaked preference domain, we show that anonymous, dominant strategy implementable, non-wasteful (a new definition we give) mechanisms are exactly *embedded generalised median voting rules*. These have a small range. We show that the difference in average ex-ante welfare of DIC mechanisms with and without restricted communication is $\Omega(k^{-2})$ for the case of two players. Furthermore, we strengthen a characterisation of anonymous strategy-proof voting schemes by Weymark 2011.

(c) We define a continuous extension of the voting model on linear tax schedules by Romer et al. 1975. We identify two properties (Properties A and B) that preference domains need to satisfy such that the characterisation we presented for single-peaked preferences also holds. We show that any regular, single-crossing, tops-connected (RST) preference domain, among them the quadratic as well as the linear preference domain, satisfies property A. We present recent evidence from Achuthankutty and Roy 2018 why property B might also be satisfied by an RST preference domain.

Literature Review

This thesis compares voting systems using ex-ante (cardinal) welfare. The idea to compare voting systems using ex-ante (cardinal) welfare goes back to Rae 1969. On the one hand, there are papers covering the case of a small number of agents or alternatives: For two alternatives and an arbitrary number of agents, Schmitz and Tröger 2012 show for a DIC setting and Azrieli and Kim 2014 for a BIC setting that interim Pareto efficient mechanisms are exactly qualified majority rules. For three alternatives and two agents, Börgers and Postl 2009 characterise BIC mechanisms in a setting where it is common knowledge that the most preferred alternative of one agent is the least preferred alternative of the other agent. On the other hand, for an arbitrary number of agents Gershkov et al. 2018 characterise in a DIC setting with a regular maximally single-crossing domain anonymous mechanisms that maximise ex-ante cardinal utility. We follow this literature in that we maximise ex-ante utility given DIC constraints.

Another branch of literature connected to ours is the approximation of mechanisms by simple mechanisms: For the case of matching, McAfee 2002 and Hoppe et al. 2011 (the second for the case with private information) compared the performance of three matching mechanisms: completely random matching, assortative matching (optimal matching given complete information revelation) and “coarse” matching, optimal matching given agents only send one of two possible messages. The papers show the coarse scheme yields at least as high welfare as the average of assortative matching’s and random matching’s ex-ante welfare. In a BIC public goods setting, Ledyard and Palfrey 2002 study the performance of mechanisms for public goods provision. They compare the optimal interim efficient mechanism to a mechanism with two messages: Agents send either one of two possible messages (for or against) to the principal. If the number of positive votes surpasses a threshold, the public good is produced and the costs are shared equally. They show that for an optimal choice of the threshold the simple mechanism's welfare converges to the welfare of the interim efficient mechanism. A similar result is the discussion at the beginning of Gershkov et al. 2017, Section 5, p. 21. The authors show that in their DIC model, a mechanism where agents send one of two possible alternatives gives the same welfare as the second-best mechanism in the limit of large societies. For the allocation of a divisible good, Wilson 1989 studies the approximation of efficient screening mechanisms in a DIC setting when agents are only allowed to send one of a finite number k of signals ("priority classes"). He shows that welfare converges to the optimal welfare with a welfare loss $O(k^{-2})$. Our work differs from this literature in that it does not consider a fixed class of mechanisms with a certain structure that approximate an optimal one, but considers a constrained optimal mechanism subject to a bound on the number of different messages an agent can send.

Our work is also connected to literature on the optimal design of signal spaces for agents: Alonso and Matouschek 2008 studies the design of the message space of one agent to a principal in the case when the utilities of the agent and the principal are misaligned. Rosar 2015 shows in a BIC setting that with an appropriately designed message space, the mean mechanism always welfare-dominates the median mechanism, exaggerated signals being allowed, but only to the extent that the message space allows. These papers consider binary resp. convex uncountable message spaces for the agents. We, however, consider message spaces of arbitrary *finite* cardinality.

There is an extensive literature on mechanism design under limited communication that allows for *monetary* transfers: Blumrosen and Feldman 2006 studies a DIC setting with single-crossing preferences, monetary transfers and a welfare function that is linear in each agent's one-dimensional type. A rate of convergence of welfare under communication constraints by $O(k^{-2})$, where k is the number of different signals an agent can send, is established and optimal mechanisms are characterised. It should be stressed that these results rely on the assumption of linear welfare functions. Blumrosen, Nisan, et al. 2007 study auction design both from a welfare maximisation perspective (with a DIC setting) as well as from a revenue maximisation perspective (with a BIC setting). They show that the strategies of agents will be partitional in the sense that agents report an interval their type lies in. Furthermore, they show that revenue converges exponentially and welfare as $O(k^{-2})$. They characterise optimal mechanisms in the special case of two players and several alternatives or two alternatives and several players. The paper Bergemann, Shen, Xu, and E. M. Yeh 2011 studies a one-dimensional screening model and gives necessary conditions for optimal revenue maximising mechanisms in a BIC mechanism as well as welfare maximising mechanisms disregarding incentive constraints. They show, as aforementioned papers, a welfare convergence rate of $O(k^{-2})$. The follow-up paper Bergemann, Shen, Xu, and E. Yeh 2012 restricts itself to the case of welfare maximisation, but considers n-dimensional types or, mathematically equivalent, n agents with one-dimensional types (but see the remark on 12). They prove rates of convergence results depending on dimension and show the suboptimality of treating each dimension of types separately in the agents' strategies. Madarász and Prat 2010 studies a one-dimensional screening problem, but is concerned with the post-correction of the effects of an approximate type space. Our work differs from this literature in that we do not allow for monetary transfers.

The paper of McMurray 2017 studies in a game-theoretic Bayes-Nash setting "mandates" for candidates in elections: Assuming candidates that maximise social welfare, agents can only cast a vote for one of the candidates. In a one-dimensional voting model, the elected candidate only chooses her ideological position after the election, and may base her decision on the election's outcome. Interpreting the choice of the ideological position of the elected candidate as a mechanism design problem, the paper gives an insight into mechanism design with limited communication: In particular, the number of votes a candidate receives can change the outcome in this BIC model. We study DIC constraints and our conclusions differ from the ones in McMurray 2017.

In the electrical engineering and statistics literature, communication under constraints is an important topic of study. There are two branches of literature particularly connected to the present study: quantisation and distributed inference.
Quantisation is the theory of the optimal approximation of a real-valued (or more general) random variable (the "signal") by a finite number of discrete levels or subject to a bound on the entropy of the discretised random variable. In our application, the further is more relevant. Lloyd 1982 and Max 1960 independently discovered a strong necessary condition for the optimal quantisation of a square-integrable random variable, whose convergence guarantee is the best known for general distributions.
Distributed Inference is the problem of estimating a quantity from observations by different sensors by a principal (the "fusion center"). The sensors send quantised messages to the principal. The principal uses a mechanism (the "fusion rule") to determine a quantity from the different messages. By treating the sensors as agents that act strategically, this is a mechanism design problem. Such cases might occur if the sensors are in public. The problem has been studied with respect to possible attacks by persons-in-the-middle, Kailkhura et al. 2015, and with algorithms to compute optimal quantisations, Venkitasubramaniam et al. 2007. From an economic perspective treating sensors as agents suggests itself and yields mechanism design problems in the special case of voting with restricted communication.

The plan of the rest of the thesis is as follows: In section 2, we present our voting model with restricted communication. We characterise in section 3 first-best mechanisms in the case of quadratic preferences and characterise convergence of average welfare. In section 4 we characterise the class of second-best mechanisms in a voting with restricted communication for the preference domain of single-peaked preferences. We give a formulation of our characterisation theorem based on two abstract properties and show that the domain restriction of RST domains implies one of them in section 5 and give a conclusion in section 6.

2 Model

Agents and Utilities We consider n agents $i = 1, 2, \ldots, n$ that have to make a choice on a continuous value of common interest. Call the *set of outcomes* $\mathcal{K}$ and the *set of types* $\mathcal{T}$ which we both assume to be totally ordered (e.g. $\mathcal{T} = \mathbb{R}$ and $\mathcal{K}$ a subset of $\mathbb{R}$). To simplify notation, we use for both total orders the symbol $<$. The set of outcomes is assumed to contain a lower bound $\underline{k}$ and an upper bound $\overline{k}$. The type set $\mathcal{T}$ consists of preference orders, i.e. reflexive, transitive, asymmetric relations over alternatives in $\mathcal{K}$. Agent $i = 1, 2, \ldots, n$ has a type $t_i \in \mathcal{T}$ privately known to her. By $\tau_{\mathcal{K}}\ (t)\ \in \mathcal{K}$ we denote the most preferred alternative of an agent with type $t \in \mathcal{T}$. We will write $x_1 \preceq^t x_2$ instead of $x_1 t x_2$ for clarity of notation. In examples, we will use utiltity functions

$$u^x \colon \mathcal{T} \to \mathbb{R}, \quad t \mapsto u^x(t),$$

for alternatives $x \in \mathcal{K}$. These induce quasi-orders, i.e. there are types t such that t is indifferent when given the choice between $x_1, x_2 \in \mathcal{K}$. We stress that there might be even several such pairs. As our model requires strict preferences, in cases where we work with utility functions, we will need to define a total order that refines this quasi-order. We call such a definition a *resolution of indifferences*. We will specify a resolution of indifferences where needed.

In the different sections, we study different domains of preferences $\mathcal{T}$. In section 3 and section 5, we consider quadratic preferences: $\mathcal{T} = \mathcal{K} = [0, 1]$ with utility functions $u^x_{\text{quad.}}(t) = -(t-x)^2 = -\|t-x\|_2^2$. As an example, consider Figure 1. The preferences of the agents are read from top to bottom, hence

$$\begin{aligned} x_1 \succ^t x_2 \succ^t x_3 \\ x_2 \succ^{t'} x_1 \succ^{t'} x_3 \end{aligned}$$

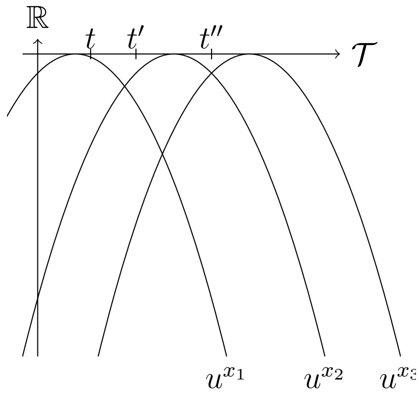


Figure 1: Quadratic preference domain

$$x_2 \sim^{t''} x_3 \succ^{t''} x_1,$$

where agent t's preference requires a resolution of indifference that is formally defined on p. 38. Informally, for each $t \in [0,1]$, we add two types $\underline{t}$, $\overline{t}$ such that in case of an indifference, $\underline{t}$ always prefers smaller outcomes, $\overline{t}$ always larger ones.

On the other hand, In section 4, we consider the set of all single-peaked preferences. A preference relation $t \in \mathcal{T}$ is *single-peaked* if for any $x_1, x_2 \in \mathcal{K}$ such that $x_2 < x_1 < \tau_{\mathcal{K}}(t)$ or $\tau_{\mathcal{K}}(t) > x_2 > x_1$ it holds that $\tau_{\mathcal{K}}(t) \succeq^t x_1 \succeq^t x_2$.

Mechanisms and Strategies A deterministic *indirect* mechanism asks agents to report one message $m \in \mathcal{M}$ where $\mathcal{M}$ is a *message set*. The mechanism then chooses an alternative from $\mathcal{K}$. Formally, an indirect mechanism is a function

$$g\colon \mathcal{M}^n \to \mathcal{K}.$$

In the following, we just write "mechanism" if there is no risk of ambiguity. We assume implicitly that participation in the mechanism is obligatory. We are interested in the case where $\mathcal{M}$ is finite, $k = |\mathcal{M}|$, hence, where the agents cannot report their complete preferences, but their report must be noisy.

We call a mechanism $g\colon \mathcal{M}^n \to \mathcal{K}$ *anonymous* if for any permutation $\pi\colon \{1,2,\dots,n\} \to \{1,2,\dots,n\}$ and any messages $m_1, m_2, \dots, m_n \in \mathcal{M}$, it holds that

$$g(m_1, m_2, \dots, m_n) = g(\pi(m_1), \pi(m_2), \dots, \pi(m_n)).$$

Less formally, this is the well-known requirement for a voting system that identity of the voters should be irrelevant for the outcome of the voting system.

A *strategy* for agent i, $i = 1, 2, \ldots, n$ is a mapping from types to reports, formally

$$s_i \colon \mathcal{K} \to \mathcal{M}, \quad i \in \{1, 2, \ldots, n\}.$$

We stress that we only allow for *pure* strategies. For strategies $s_1, s_2, \ldots, s_n$ $i = 1, 2, \ldots, n$, and $t_1, t_2, \ldots, t_n, t_i' \in \mathcal{T}$, we will use the notation $(s_i(t_i'), s_{-i}(t_{-i}))$ for

$$(s_1(t_1), s_2(t_2), \ldots, s_{i-1}(t_{i-1}), s_i(t_i'), s_{i+1}(t_{i+1}), \ldots, s_n(t_n)) \in \mathcal{M}^n$$

A mechanism g is said to be *implementable in dominant strategies* by strategies s_1, s_2, $\ldots$, s_n if for any $m_{-i} \in \mathcal{M}^{n-1}$ if for any $i = 1, 2, \ldots, n$

$$g(s_i(t_i), m_{-i}) \succeq^{t_i} g(m, m_{-i}). \tag{1}$$

Less formally, $s_i(t_i)$ must be a best response given any other messages the other agents send.

Finally, one might want to require efficiency of the mechanism.[1] We assume two efficiency requirements, one concerning the mechanisms, one concerning strategies. The first is needed to rule out different strategies yielding an ex-post identical outcome. A mechanism is *non-wasteful* if

$$\text{for any } m, m' \in \mathcal{M} \text{ there is } m_{-i} \in \mathcal{M}^{n-1} \text{ such that } g(m, m_{-i}) \neq g(m', m_{-i}) \tag{2}$$

The second assumption requires that for the mechanism the size k of the message set is actually needed: For any message, there is a type that plays it.[2]

$$\text{all } s_i \colon \mathcal{T} \to \mathcal{M} \text{ are surjective} \tag{3}$$

[1]It is unreasonable to assume unanimity or the even stronger property of Pareto optimality of the mechanism, i.e. the requirement that if all agents have a common most preferred alternative that the mechanism should implement this alternative, as this leaves no incentive compatible mechanisms as soon as there are more types with different most preferred alternatives than messages k in the message set $\mathcal{M}$. Furthermore, the requirement that the outcome should be unanimous w.r.t. the preferences on the range of the social choice function immediately yields a characterisation result: In this case, the messages must communicate the top alternative among the alternatives in the range and the range must be smaller or equal than k.

[2]Both (2) and (3) are requirements of not wasting communication resources. We separate the two to stress that one is a requirement on the mechanism, the other a requirement on the strategies.

Type Distribution and Welfare Maximisation For later welfare maximisation, we assume that $t_i \sim F$, $i = 1, 2, \ldots, n$ are identically and independently distributed according to a cumulative distribution function F. This is a classical setting for voting as studied e.g. in Gershkov et al. 2018.

We call

$$W(F, g) \coloneqq \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} u^{x_i}(g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)))\right]$$

the *average ex-ante welfare* of the mechanism g. The factor $\frac{1}{n}$ is irrelevant for later utility maximisation, but important for asymptotic results. A mechanism that is ex-ante welfare maximising is said to be *first-best*. A mechanism that is ex-ante welfare maximising among all anonymous, non-wasteful mechanisms that are dominant strategy implementable by surjective strategies is called *second-best*. This is non-standard terminology. It would be more specific to write constrained first-best with respect to restricted communication. For the sake of brevity and as we do not cover first- or second-best mechanisms without restricted communication, we choose to just write first- and second-best.

Notions from the literature on voting with unrestricted communication In a setting, where $\mathcal{M} = \mathcal{T}$, there is enough capacity for the agents to communicate their full type, i.e. the revelation principle holds for such mechanisms. We will need some terminology from this literature. We call functions $f\colon \mathcal{T}^n \to \mathcal{K}$ *social welfare functions* and say that a mechanism $g\colon \mathcal{M}^n \to \mathcal{K}$ and strategies $s_i\colon \mathcal{T} \to \mathcal{M}$ *implement* f if

$$f(t_1, t_2, \ldots, t_n) = g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n))$$

If $s_i\colon \mathcal{T} \to \mathcal{M} = \mathcal{T}$ can be chosen to be the identity, then g is said to be *strategy-proof*. With unrestricted communication, a mechanism is dominant strategy implementable if and only if it is strategy-proof. Furthermore, we call a mechanism *unanimous* if $\tau_{\mathcal{K}}(t_1) = \tau_{\mathcal{K}}(t_2) = \cdots = \tau_{\mathcal{K}}(t_n)$ implies $g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) = \tau_{\mathcal{K}}(t_1)$.

In a setting with $\mathcal{M} = \mathcal{K} = \mathcal{T} = \mathbb{R}$ (full communication) and quadratic preferences, it is well known that the first-best mechanism, the welfare maximising decision rule disregarding incentives is the mean and the second-best mechanism is given by so-called *generalised median voting rules*. We will show that there are generalisations of both results to the present model. We start with the characterisation of first-best mechanisms.

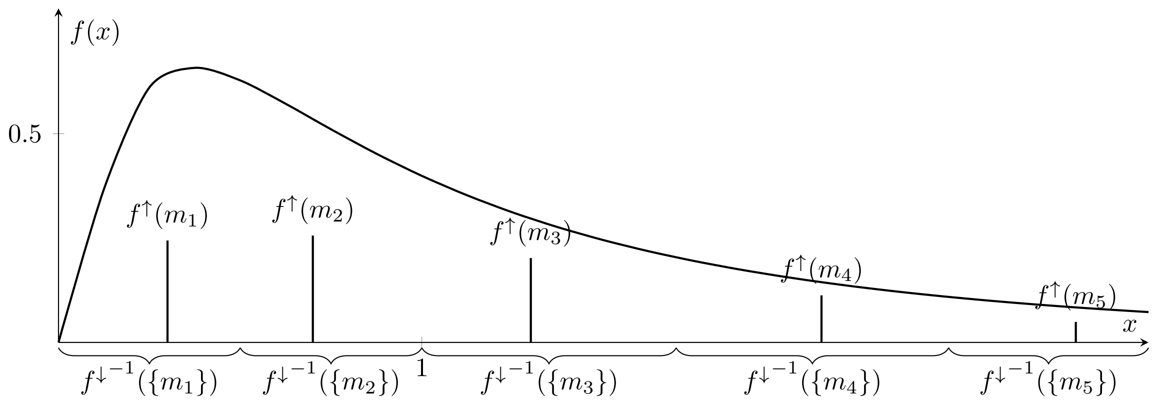


Figure 2: Example of an $\mathcal{M}$-quantisation for a log-normal distribution ($\mu = 0$, $\sigma = 1$).

3 Characterisation and Welfare Loss of First-Best Mechanisms

In this section, we consider quadratic utility functions $u^{t_i}(x) = -(t - x)^2$ and $\mathcal{T} = \mathcal{K} = [0, 1]$.[3] We will characterise the first-best mechanisms and give a tight characterisation of convergence of ex-ante average welfare.

We first need the notion of an $\mathcal{M}$-quantisation of a distribution F.

Definition. *Let $F \colon \mathbb{R} \to [0, 1]$ be a distribution and $X \sim F$. Then an $\mathcal{M}$-quantisation of F is a pair of functions $(f^{\downarrow}, f^{\uparrow})$, $f^{\downarrow} \colon \mathbb{R} \to \mathcal{M}$, $f^{\uparrow} \colon \mathcal{M} \to \mathbb{R}$. For square-integrable F, $(f^{\downarrow}, f^{\uparrow})$ is called* optimal *if it minimises*

$$\mathrm{MSE}(F, (f^{\downarrow}, f^{\uparrow})) \coloneqq \mathbb{E}[\|X - f^{\uparrow}(f^{\downarrow}(X))\|_2^2] = \mathbb{E}[(X - f^{\uparrow}(f^{\downarrow}(X)))^2].$$

with respect to $f^{\downarrow} \colon \mathbb{R} \to \mathcal{M}$ and $f^{\uparrow} \colon \mathcal{M} \to \mathbb{R}$. Call the minimiser $\mathrm{MSE}^*(F)$.

We also say that X is quantised to *quantisation level* $f^{\downarrow}(X)$ with representative point $f^{\uparrow}(f^{\downarrow}(X))$.

An example of a quantisation is shown in Figure 2. The log-normal distribution, that is sometimes used to model income distributions, is approximated by a linear combination of point measures, here denoted by bars. The function $f^{\downarrow}$ can be seen as partitioning the positive real line into intervals via their preimages, here denoted by braces. Values within an interval are all mapped to the same value m_i, that is mapped to the representative point $f^{\uparrow}(m_i)$.

[3]The choice of the unit interval is merely for notational convenience – any other compact interval would allow for similar results as the ones obtained here.

There is a well-known necessary condition for the optimal $\mathcal{M}$-quantisation, Lloyd 1982; Max 1960.

Proposition (Lloyd 1982, Eqn. (16) and (17)). *The optimal $\mathcal{M}$-quantisation satisfies the following two conditions:*

$$f^{\downarrow}(x) = \arg\min_{m \in \mathcal{M}} (x - f^{\uparrow}(m))^2 \tag{4}$$
$$f^{\uparrow}(m) = \mathbb{E}_{X \sim F}[X|\ X \in (f^{\downarrow})^{-1}(\{m\})] \tag{5}$$

The resulting quantisation is also called Lloyd-Max quantisation.

Equation (4) says that each value shall be mapped to the closest representative point of any quantisatin level, equation (5) says that the representative point of a quantisation level shall be the centroid of all points that are mapped to this level. Optimal quantisations can be computed via Lloyd's algorithm Lloyd 1982.

The main result of this section uses quantisation to give a strong necessary condition for first-best mechanisms with restricted communication:

Theorem 1 (Classification of first-best mechanisms). *Let F be a square-integrable,* $[0, 1]$-*valued distribution with optimal $\mathcal{M}$-quantisation $(f^{\uparrow}, f^{\downarrow})$ and variance σ^2. Then there is a first-best mechanism $g_{\text{first-best}}$ together with implementing strategies s_1, s_2, $\ldots$, s_n such that*

$$g_{\text{first-best}}(m_1, m_2, \ldots, m_n) = \frac{1}{n} \sum_{i=1}^{n} f^{\uparrow}(m_i)$$
$$s_i(x) = f^{\downarrow}(x).$$

In addition, $-W(F, g_{\text{first-best}}) = \frac{1}{n} \operatorname{MSE}^*(F) + \frac{n-1}{n}\sigma^2$.

In other words, the same quantisation is applied to each agent's type separately and the reported representative points are averaged.

In the case of unrestricted communication, the best average welfare is obtained by the mean mechanism, the social choice function $f \colon \mathcal{T}^n \to \mathcal{K}, (t_1, t_2, \ldots, t_n) \mapsto \frac{1}{n} \sum_{i=1}^{n} t_i$. It obtains an average welfare of $W(F, f) = -\frac{n-1}{n}\sigma^2$. Therefore, it is fair to consider $-W(F, f) - \frac{n-1}{n}\sigma^2 \geq 0$ for average welfare comparison. We show quadratic convergence of average welfare in k, the number of signals an agent can send, matching existing results in the literature on mechanism design with monetary transfers Bergemann, Shen, Xu, and E. M. Yeh 2011; Blumrosen, Nisan, et al. 2007 and linearly in the size of the society n.

Proposition 2 (Convergence of average welfare for first-best mechanism)**.** *Let F be a distribution on $[0,1]$ with variance σ^2. For any $\mathcal{M}$ with $|\mathcal{M}| = k$, we have*

$$\frac{1}{12nk^2} \leq -\frac{n-1}{n}\sigma^2 - W(F, g_{\text{first-best}}) \leq \frac{1}{4nk^2}, \tag{6}$$

hence $\frac{n-1}{n}\sigma^2 + W(F, g_{\text{first-best}}) \in \Theta(k^{-2}) \cap \Theta(n^{-1})$.

Note that the loss due to quantisation is relatively minor: If k is kept fixed, then considering the limit in n, the welfare loss of the first-best mechanism without communication restrictions (an average welfare of zero would mean that everyone has assigned exactly her type) is of another order of magnitude as the difference in welfare between the first-best mechanism with unrestricted communication and the first-best mechanism.

Remark. *In light of Bergemann, Shen, Xu, and E. Yeh 2012, Section 2.2, one might ask why there is no advantage of vector quantisation in this model, i.e. why it is welfare maximising to treat each type seperately. This is based on a problem in the model of Bergemann, Shen, Xu, and E. Yeh 2012. The authors note that the problems of multi-product and multi-agent auctions are* mathematically equivalent*. In a setting that the paper Bergemann, Shen, Xu, and E. Yeh 2012 models, they are likely not: If vector quantisation is used, then the mechanism does not depend separately on each type, but the mechanism depends on all types lying in a complex set in $\mathcal{T}^n$. To decide in which of these regions the agents' types lie in, the agents have to communicate their type to each other—likely not possible in a setting of restricted communication.*

Finally, we consider the cardinality of the range of first-best mechanisms. Even with a trivial lower bound we can show that the range grows at least linearly, despite the communication of agents being bounded. Although this result is tight, we believe that generically in the set of measures $\mathcal{M}([0,1])$ much better lower bounds hold.[4]

Corollary 3 (Range of first-best mechanisms)**.** $|\text{range}\, g_{\text{first-best}}| \geq (k-1)n + 1$ *and this bound is tight.*

In this section, we hence saw that first-best mechanisms are anonymous, their average welfare converges quadratically in k and linearly in n and their range grows at least linearly both in k and n.

[4]The proof of the following result(s) can be found in the appendix.

4 Characterisation and Welfare Loss of Incentive Compatible Mechanisms

In many situations, the first-best is not implementable in dominant-strategies. This occurs even for $|\mathcal{M}| = 3$.

Example. *Consider quadratic utilities* $u^x \colon \mathcal{T} \to \mathbb{R}_+, u^x(t) = -(x-t)^2$. *Let* F *be the uniform distribution on* $[0,1]$ *and let* $\mathcal{M} = \{1,2,3\}$. *Then the Lloyd-Max conditions* (4) *and* (5) *determine the optimal* $\mathcal{M}$ *quantiser up to enumeration of* $\mathcal{M}$ *and* F*-zero sets. It is called the* uniform quantiser *(cf. Bergemann, Shen, Xu, and E. M. Yeh 2011, Example 1):*

$$f^{\downarrow}(x) = \begin{cases} 1 & \text{if } x \in [0, \frac{1}{3}] \\ 2 & \text{if } x \in (\frac{1}{3}, \frac{2}{3}] \\ 3 & \text{if } x \in (\frac{2}{3}, 1] \end{cases} \qquad f^{\uparrow} \begin{cases} 1 \mapsto \frac{1}{6} \\ 2 \mapsto \frac{3}{6} \\ 3 \mapsto \frac{5}{6} \end{cases}$$

Hence, by Theorem 1 the first-best mechanism resp. implementing strategies are

$$g_{\text{first-best}}(m_1, m_2, \ldots, m_n) = \frac{1}{6}m^1 + \frac{1}{2}m^2 + \frac{5}{6}m^3 \qquad s_i(t) = f^{\downarrow}(t)$$

where $m^j = \frac{1}{2}\left|\{i = 1,2,\ldots,n \mid m_i = j\}\right|$. *Problematically, agent* i *can profitably deviate from strategy* s_i*: Assume* $t_i \in (\frac{1}{2}, \frac{2}{3})$ *and let* $m^1 > m^3$. *In this case,* $g_{\text{first-best}}(1, m_{-i}) < g_{\text{first-best}}(2, m_{-i}) < g_{\text{first-best}}(3, m_{-i}) < \frac{1}{2} < t_i$. *Hence,* i *can deviate by sending* 3 *instead of* 2.

The observation that the possibility to announce extreme alternatives breaks incentive compatibility has been the starting point of the voting literature, Galton 1907. As this is present also in our model, we continue to characterise dominant strategy implementable mechanisms—for tractability on the domain of single-peaked preferences.

Generalised median voting schemes are well-known in the literature on strategyproof social choice. They have been shown to be exactly the unanimous, anonymous, strategyproof mechanisms Moulin 1980.

Definition. *A* generalised median voting scheme *is a social choice function* $f \colon \mathcal{T}^n \to \mathcal{K}$ *such that there are* $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \mathcal{K}$ *(the* phantom ballots*) such that*

$$g(t_1, t_2, \ldots, t_n) = \operatorname{med}\{\tau(t_1), \tau(t_2), \ldots, \tau(t_n), \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\},$$

where $\operatorname{med} A$ *for* $A \subseteq \mathcal{K}$ *is the median with respect to the order on* $\mathcal{K}$.

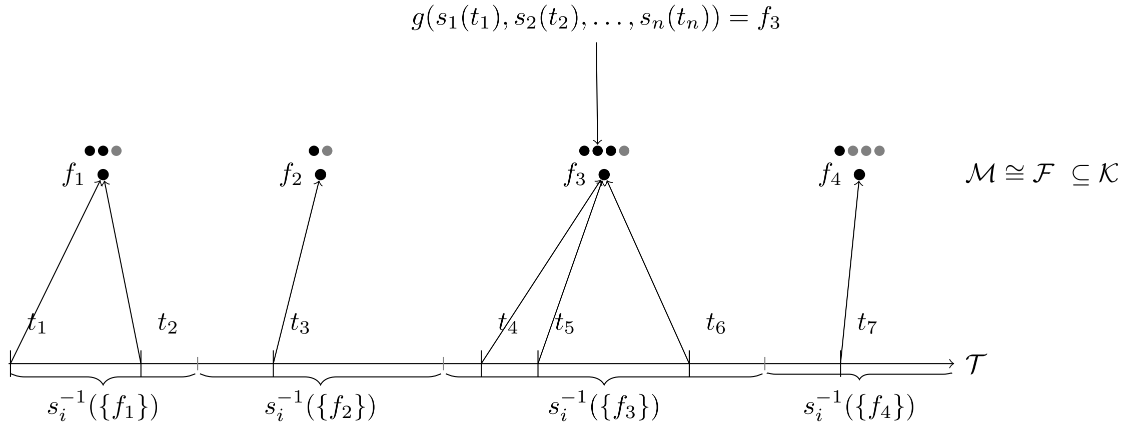


Figure 3: Embedded generalised median voting rules

We define a communication-restricted variant of generalised median voting rules: embedded generalised median voting rules.

Definition. *An* embedded generalised median voting rule *is a mechanism* $g\colon \mathcal{M}^n \to \mathcal{K}$ *if there is an injective function* $\iota\colon \mathcal{M} \to \mathcal{K}$ *(the* embedding*) and* $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \operatorname{range} \iota$ *(the* phantom ballots*) such that*

$$g\colon \mathcal{M}^n \to \mathcal{K} g(m_1, m_2, \ldots, m_n) = \operatorname{med}\{\iota(m_1), \iota(m_2), \ldots, \iota(m_n), \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}.$$

Figure 3 illustrates generalised median voting rules. As for the first-best mechanisms, the strategies s_i yield a partition of the type space that is illustrated by braces. There is a bijective mapping between the message space $\mathcal{M}$ and the range $\mathcal{F}$ of the mechanism g. The agents send messages m that are interpreted as the representative point $\iota(m)$ depending on which set in the partition they lie in (upwards arrows). The votes for the different values are interpreted as ballots in a generalised median voting rule (black bullets). Together with phantom ballots (grey bullets) that can only lie on representative points, the median is implemented (downwards arrow).

The following is the main result of this section. It shows that a restriction on the communication of agents to the principal rules out all mechanisms but embedded generalised median voting rules.

Theorem 4 (Classification of dominant incentive compatible anonymous non-wasteful mechanisms, single-peaked)**.** *Let* $\mathcal{T}$ *be the set of single-peaked preferences on* $\mathcal{K}$*. Then* $g\colon \mathcal{T}^n \to \mathcal{K}$ *is anonymous, non-wasteful and DIC implementable by surjective strategies if and only if it is an embedded generalised median voting rule.*

This theorem is surprising in several respects: First, it shows that a communication bound on the messages that agents can send in a DIC regime leads endogenously to an embedding of the message set into the choice set and hence that the optimisation problem for the second-best mechanism consist of optimising the positions of both representative points and phantom ballots. Given the positions of the representative points, analytic solutions for the distribution of phantom ballots in a second-best mechanism has been given in Gershkov et al. 2017 under mild conditions on the type distribution.

Second, the theorem says that a bound on the communication implies that the communication from the principal to the agents will be limited: k values in $\mathcal{K}$ numbers (the positions of the representative points) and k integers of size smaller than n (the numbers of phantom ballots on each representative point). An immediate corollary of Theorem 4 is:

Corollary 5 (Range of second-best mechanism)**.** *The second-best mechanism has a range of cardinality at most k.*

Third, Theorem 4 can be seen as evidence for the prevalence of parties in representative systems. For example, in Germany, the constitutional foundation of elections of the German parliament is given in Article 38 of the German *Grundgesetz* with no reference to parties. Only the election of representatives is considered. Only in article 21, parties are mentioned, with no reference to elections, but with reference to formation of the political will ("politische Willensbildung"). Theorem 4 says that when the communication of voters is limited, e.g. due to intellectual limits of voters in evaluating a large number of different positions, then a small number of "parties" with representative points on the political spectrum will form and votes are taken among parties subject to qualified majority requirements or equivalently generalised median voting rules (compare Barberà 2001, p. 630 or Gershkov et al. 2017, Section 3).
The DIC requirement in Theorem 4 is a robustness requirement that only holds if the beliefs of agents are precarious. In particular, it is to be expected that the range of mechanisms with weaker assumptions on incentive compatibility such as BIC might be larger.

Fourth, this theorem implies that dominant strategy implementability might mean a large welfare loss. The following contribution shows that embedded generalised median voting rules are too restricted to guarantee a quadratic convergence of welfare in k.

As in the case of the first-best mechanism, it is a question what to compare the welfare of an embedded generalised median voting rule to. One should disentangle three sources of welfare loss: The first loss of $\frac{n-1}{n}\sigma^2$ is due to the fact that the agents need to find one common value to implement. The second is the welfare loss due to incentive compatibility. The third is the loss due to quantisation. To get the effect of quantisation within incentive compatible models, we compare the welfare of any generalised median voting rule, one of which is the second-best mechanism in a model without communication restrictions, to any embedded generalised median voting rule.

For tractability, we assume quadratic utilities. Confirming intuition, the welfare maximising embedded generalised median voting rule converges in welfare to the welfare maximising DIC mechanism without communication restrictions. [5]

Proposition. *Let g be a welfare maximising unanimous, anonymous, strategy-proof mechanism g in a model without communication restriction, i.e. a generalised median voting rule and F any distribution on $[0,1]$. Then there is a sequence $(g_k)_{k\in\mathcal{N}}$ of embedded generalised median voting rules $g_k\colon \mathcal{M}^n \to \mathcal{K}$ with $|\mathcal{M}| = k$ such that*

$$W(F,g) - W(F,g_k) \in o(1).$$

On the other hand, even for $n = 2$, this convergence is strictly slower than the convergence of the first-best mechanism to the first-best in a model without communication restrictions.[6]

Proposition 6 (Welfare comparison for second-best mechanism)**.** *Assume quadratic preferences and $n = 2$. There is a distribution F such that the following holds:*

Let g be any unanimous, anonymous, strategy-proof mechanism g in a model without communication restriction, i.e. a generalised median voting rule, and let $(g_k)_{k\in\mathcal{N}}$ be any sequence of embedded generalised median voting rules $g_k\colon \mathcal{M}^n \to \mathcal{K}$ with $|\mathcal{M}| = k$. Then

$$W(F,g) - W(F,g_k) \in \Omega(k^{-1}).$$

where Ω is the Landau symbol for at most as fast convergence.

Hence, the welfare loss due to quantisation is larger in the case of DIC mechanisms than in the first-best case.

We remark that it is not too hard to get upper bounds on the welfare of g_k as opposed to the difference in welfare to generalised median voting rules—the kind of result we gave above: As g_k takes only k values, the bound in Bergemann, Shen, Xu, and E. Yeh 2012, Lemma 1 might be employed if one recognises that social welfare functions implemented

[5] The proof of the following result(s) can be found in the appendix.
[6] The proof of the following result(s) can be found in the appendix.

by indirect mechanisms and strategies are examples of vector quantisers, cf. Bergemann, Shen, Xu, and E. Yeh 2012. Unfortunately, application of their result does not give a general lower bound when comparing embedded generalised median voting rules to generalised median voting rules or the mean rule.

We close this section with a strengthening of Weymark 2011, Theorem 4 for non-interval ranges of the range of the social choice function, which is proved similar in spirit as Theorem 4.

Proposition 7 (Characterisation of strategyproof anonymous mechanisms for non-interval ranges)**.** *Let $\mathcal{T}$ be set of single-peaked preferences. Let $f : \mathcal{T}^n \to \mathcal{K}$ be a social choice function. Then, f is strategy-proof and anonymous if and only if there there are $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that for*

$$f'\colon \operatorname{range} f \to \operatorname{range} f, \quad f'(t_1, t_2, \ldots, t_n) = \operatorname{med}\{t_1, t_2, \ldots, t_n, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \tag{7}$$

it holds that

$$f(t_1, t_2, \ldots, t_n) = f'(\tau_{\operatorname{range} f}(t_1), \tau_{\operatorname{range} f}(t_2), \ldots, \tau_{\operatorname{range} f}(t_n)).$$

One might wonder why there is no $(n+1)$-parameter generalised median voting rule involved. This is due to the requirement that f' must be surjective by definition and Lemma 1, that is proved on p. 31 in the appendix.

This result says that DIC anonymous mechanisms might pool agents with types adjacent in $\mathcal{T}$ in a way that agents report their top preference in range(f) and then a generalised median scheme is applied. Hence the mechanisms are not tops-only in general but tops-only on the range of the mechanism.

5 A Preference Restriction for Voting with Restricted Communication

The Sections 3 and 4 are disconnected in the sense that the domain of quadratic preferences is strictly smaller than the set of single-peaked preferences.

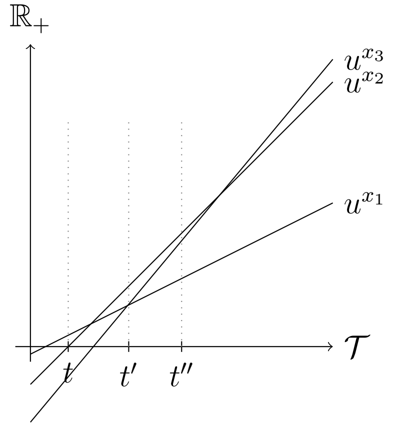


Figure 4: Linear preference domain

Example. *The quadratic voting model does not induce the complete set of single-peaked preferences. Indeed, the utility function*

$$u^x(t) = \begin{cases} x - t & x < t \\ -(x-t)^2 & x \geq t \end{cases}$$

induces a single-peaked preference with peak $x \in \mathcal{K}$*. Any quadratic utility is symmetric in the sense that if* $x_1 < x_2 \in \mathcal{K}$*,* $x + \varepsilon \succeq^{\frac{x+y}{2}} y$ *and* $y + \varepsilon \succeq^{\frac{x+y}{2}} x$ *for any* $\varepsilon > 0$*. But for the preference induced by* u^x*, this holds only if* $x - y = 2$*.*

One should aim for results for small preference domains, as in many models, it might contradict fundamentals of the model if agents are allowed to report too many preferences.

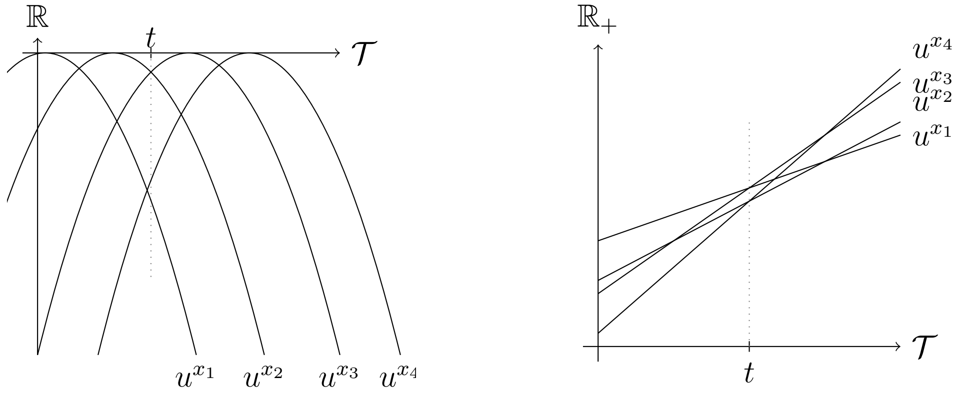
To show that the generality of the preference domain we are going to define, we present a continuous variant of the linear preference domain from Romer et al. 1975.

Definition (Linear Preferences)**.** *Let* $a\colon [0,1] \to \mathbb{R}$ *be a strictly decreasing and* $b\colon [0,1] \to \mathbb{R}_{\geq 0}$ *be a strictly increasing, nonnegative function. Consider the utility functions* $u^x_{lin.}(t) = a(x) + b(x)t$*.*

The model has been used to model elections on linear tax schedules. $u^x_{\text{lin.}}(t)$ is the post-tax income for productivity type t under tax schedule choice x.

Figure 4 shows the linear model. Each type's preference is given from the highest utility to the lowest. The preferences of the agents t, t' and t'' are

$$x_1 \succ^t x_2 \succ^t x_3$$



(a) Quadratic case

(b) Linear case

Figure 5: Indifferences.

$$x_2 \succ^{t'} x_1 \sim^{t'} x_3$$
$$x_2 \succ^{t''} x_3 \succ^{t''} x_1$$

As in the quadratic environment, the preference of agent t' depends on how indifferences are resolved. The informal construction on p. 7 that is formally given on 38 suuffices to give a sufficiently rich domain of preferences.

As the quadratic case, this model does not induce all single-peaked preferences. Indeed, the example in Gershkov et al. 2017, p. 7f. for a finite choice set $\mathcal{K}$ can be applied here as well.

After having defined two utility models for voting with restricted communication, we continue with the definition of a sufficiently general preference restriction that generalises the two. Candidates are maximally single-crossing domains. On these, strong characterisation results for strategy-proof mechanisms have been proved. In the following definition, we use the notation $\mathcal{T}_A = \{t|_{A\times A} \mid t \in \mathcal{T}\}$ for the preferences in $\mathcal{T}$ restricted to the set $A \subseteq \mathcal{K}$.
However these domains are required to be too large to be useful for the study of mechanisms of restricted communication. The reason are intersection points of utility functions: If a type has several crossings of utility functions (and this number is not necessarily bounded, neither for the quadratic nor for the linear model), then a simple resolution of indifferences as we proposed will mean that adjacent types might have different relative rankings of more than two alternatives which is not permitted by maximality (compare Gershkov et al. 2017, Fn. 28). More complicated indifference-resolution schemes are harder to be shown to be single-crossing, if even existent. For the quadratic model, the property that all preferences on $\mathcal{T}_A$, $A \subset \mathcal{K}$ finite, do not contain any crossing point is

generic. However, to find a second-best, the set A is itself optimised and non-generic outcomes might be maximisers. Therefore, we consider a domain restriction that allows for multiple crossing points on $\mathcal{T}_A$ with our indifference resolution scheme.

Definition. *For a totally ordered set $\mathcal{T}$ of preference relations on $\mathcal{K}$ we define:*

(a) $\mathcal{T}$ is called regular domain *if for any $k \in \mathcal{K}$ there is $t \in \mathcal{T}$ for which $k = \tau_{\mathcal{K}}(t)$.*

(b) $\mathcal{T}$ is called single-crossing *if $k < k' \ \in \mathcal{K}$, $t < t' \ \in \mathcal{T}$ $k \preceq^t k'$, then $k \preceq^{t'} k'$.*

(c) $\mathcal{T}$ is called tops-connected if for any finite set $A \subseteq \mathcal{K}$, and adjacent $x, y \in A$, i.e. there is no $z \in A$ such that $x < z < y$ or $x > y > z$ there is $t_A \in \mathcal{T}_A$ such that $\tau_A(t_A) = x$, $\tau_{A\setminus\{x\}}(t_A) = y$.

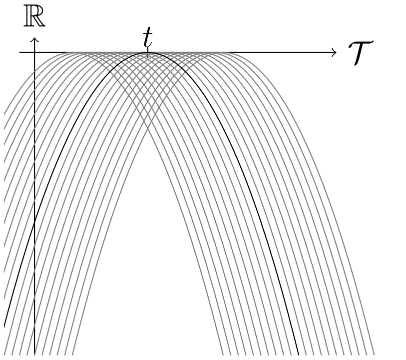
Regularity, single-crossing and tops-connectedness are depicted in Figure 6 for both the quadratic and the linear preference domain. A preference domain is regular if the function $\tau_{\mathcal{T}} : \mathcal{T} \to \mathcal{K}$ is surjective, hence if there are no alternatives that are never the most-preferred alternatives. In the quadratic case, this holds unconditionally (compare Figure 6a) and in the linear case conditional on u^x touching $\max_{t\in\mathbb{R}} u^x(t)$ (compare Figure 6b). Single-crossingness is the requirement that if one type prefers a larger alternative to a smaller, then so do all larger types as well (compare Figure 6c and Figure 6d). If the preferences are represented by utility functions, this is the requirements that the utility functions only cross a single time. Tops-connectedness is perhaps the least well known notion of the three. It says if we consider any finite subset (e.g. $\{x_1, x_2, x_3, x_4\} \subseteq \mathcal{K}$), then for adjacent alternatives such as x_1 and x_2, there is a type $t \in \mathcal{T}$ such that x_2 is the first and x_3 the second. In Figure 6e and Figure 6f, this is the case for all types $t \in [t', t]$.

A first reason why this preference restriction might be a sensible domain for the study of communication-constrained voting is that that the linear and quadratic preference domains are RST domains under intuitive conditions.

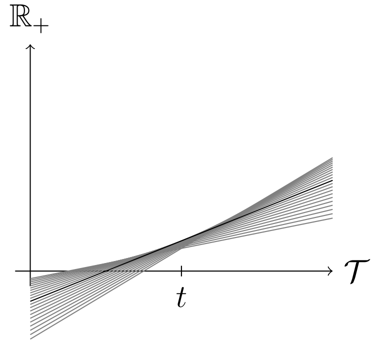
Proposition 8 (Linear and quadratic preferences domains are RST). *(a) The quadratic domain is an RST domain.*

(b) The linear preference domain is an RST domain if and only if $(u^x)_{x\in[0,1]}$ are the sub-differentials of a convex function.

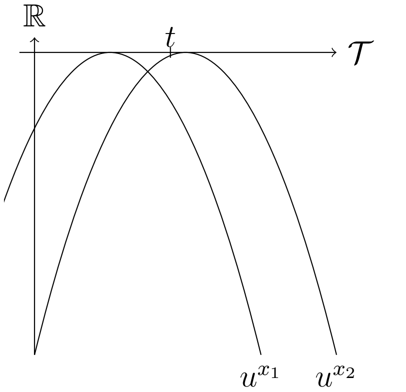
The requirement that the u^x should form the tangents of a convex function is that if agents choose their most preferred tax schedule, the after-tax income should grow super-linearly in the productivity type, which is fair to assume.



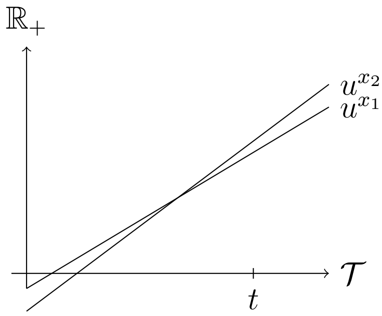
(a) Regularity (quadratic utilities)



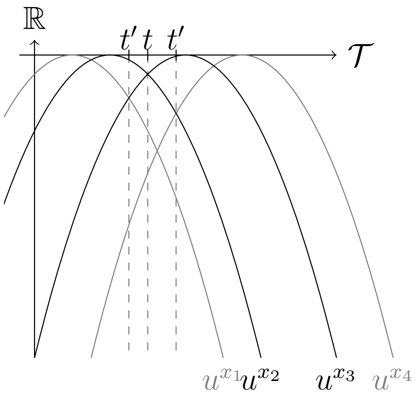
(b) Regularity (linear utilities)



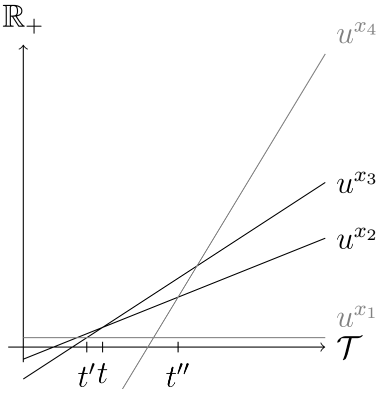
(c) Single-crossing (quadratic utilities)



(d) Single-crossing (linear utilities)



(e) Top-connectedness (quadratic utilities)



(f) Top-connectedness (linear utilities)

Figure 6: Regularity, single-crossing and tops-connectedness for quadratic and linear preference domains.

The second reason is that Theorem 4 holds more generally under two conditions. Property A requires the mapping types to tops to be monotonic and surjective. Property B requires a characterisation of strategyproof, anonymous, surjective social choice functions.

Definition (Properties A and B). *Let $\mathcal{T}$ be a preference domain. It is said to have* property A *if for any subset $\mathcal{T}$ of A, $\tau_A \colon \mathcal{T}_A \to A$ is monotone and surjective.*[7]

It is said to have property B *if for any finite $A \subseteq \mathcal{K}$ a social choice function $\mathcal{T}_A^n \to A$ is strategy-proof, surjective and anonymous if and only if it is an generalised median voting rule on $\mathcal{T}_A$.*

The significance of the two properties lies in the fact that they suffice to make the statement of Theorem 4 hold.

Theorem 9 (Classification of dominant incentive compatible anonymous non-wasteful mechanisms, axiomatic). *Let $\mathcal{T}$ be a preference domain that satisfies properties A and B. Then $g \colon \mathcal{T}^n \to \mathcal{K}$ is anonymous, non-wasteful and DIC implementable by surjective strategies if and only if it is an embedded generalised median voting rule.*

Property A is relatively easy to prove for RST domains.[8]

Proposition (RST domains: property A). *RST domains satisfy property A.*

We conjecture that RST domains also satisfy property B.

Conjecture (RST domains: property B). *RST domains satisfy property B.*

As evidence that this conjecture, we show a very similar in that unanimity instead of surjectivity is required. Using Lemma 1 and Proposition 11, which might be of independent interest and which is stated below, this evidence is a corollary of Achuthankutty and Roy 2018, Corollary 6.2.

Corollary 10 (Anonymous version of Achuthankutty and Roy 2018, Corollary 6.2). *Assume that $\mathcal{T}$ is an RST domain. Then for any finite $A \subseteq \mathcal{K}$, a social choice function $\mathcal{T}_A^n \to A$ is strategy-proof, surjective and anonymous if and only if it is an generalised median voting rule on $\mathcal{T}_A$.*

[7]Note that to say that τ_A is surjective is to say that $\mathcal{T}_A$ is a regular domain.
[8]The proof of the following result(s) can be found in the appendix.

To strengthen Theorem 4, we would need a stronger statement which replaces unanmimity by surjectivity. We leave the proof of for further work.

To close this section, we state an independent formulation of a part of the proof of Weymark 2011, Theorem 4. Recall that a max-min social choice function is defined by numbers $\{b_S\}_{S\subseteq\{1,2,\dots,n\}} \subset \mathcal{K}$ such that $b_S \leq b_T$ for any $S \subseteq T$. Then define

$$g_{\text{min-max}}(t_1, t_2, \dots, t_n) \ = \min_{S\subseteq\{1,2,\dots,n\}} \max_{i\in S}\{\tau_{\mathcal{K}}(t_i), a_s\}.$$

Min-max social choice functions have a very clear interpretation in terms of successive voting procedures, compare Barberà 2001, p. 630.

The following result characterises anonymous min-max social choice functions.[9]

Proposition 11 (Anonymous min-max is generalised median)**.** *Let $\mathcal{T}$ be regular. Then a min-max social choice function is anonymous if and only if it is an $(n+1)$-parameter generalised median voting scheme.*[10]

Similar formulations are possible for Weymark 2011, Theorems 3 and 5.

The present section might raise the interest in obtaining further classification results for strategy-proof, surjective mechanisms as these imply a strengthening of Theorem 4.

6 Conclusion

The present thesis studied information-constrained voting. The main difference to other models of voting is that the assumption of abundant capacity of communication from the agents to the principal is dropped. The number of messages agents are allowed to send is limited and indirectly dominant strategy implementable mechanisms are considered. No revelation principle can be used in this setting. Our contributions are threefold:
First, we characterised the first-best mechanism in a model of voting with quadratic preferences. We found: The first-best mechanism is automatically anonymous and agents' strategies are partitional in the sense that agents report an interval in which their type lies. The intervals are given by an optimal quantisation of the common type distribution, which can be computed tractably due to a strong necessary condition, the Lloyd-Max condition. Given the reported intervals, the first-best mechanism implements the weighted average of the interval centroids according to how many agents reported each interval.

[9]The proof of the following result(s) can be found in the appendix.
[10]For the definition of $(n+1)$-parameter generalised median voting rules see p. 31.

The interval centroids can be interpreted as ideological centers of groups of voters or members of a party. Furthermore, we show a tight bound on the growth of the cardinality of the range of a first-best mechanism.
Second, we characterise the class of non-wasteful, anonymous mechanisms that are dominant strategy implementable by surjective strategies on the preference domain of single-peaked preferences. We show that these mechanisms are exactly *embedded generalised median voting rules.* The strategies implementing these rules are partitional as in the first-best case, however, even in the quadratic voting model, the representative points in the intervals that are aggregated need not be interval centroids anymore. Aggregation of the representative points is according to a generalised median voting rule. In particular, if aggregation is done instead by the mean by a generalised median voting scheme, a mechanism implementable by the same strategies *in dominant strategies* is obtained. The choice of representative points might however not be welfare maximising. One can conclude that incentives change party positions. In contrast to the first-best case, we obtain that the cardinality of the range of non-wasteful, anonymous mechanisms that are dominant strategy implementable by surjective strategies is at most the number of signals each agent might send.
Finally, we compare the welfare loss due to quantisation for the class of all indirect mechanisms and embedded generalised median voting rules (that were shown to be the only DIC mechanisms on the domain of single-peaked preferences). We show that the welfare loss due to quantisation for the case of indirect mechanisms (hence without incentive constraints) converges in the size k of the message set as $\Theta(k^{-2})$. In contrast, for quadratic preferences there is a distribution such that the welfare difference of any generalised median voting rule and any embedded generalised median voting rule converges at most linearly, $\Omega(k^{-1}$ even in the case of two voters, $n = 2$. This is evidence that the welfare loss due to restricted communication might be particularly large when additionally requiring incentive compatibility.

quantify the welfare gap between first- and second-best mechanisms in the quadratic voting model. The average welfare loss compared to the welfare of the agents' most preferred alternative converges quadratically in the number of messages agents can send and is constant in the size of the society. There are however distributions such that the average welfare deteriorates for large societies. This shows that in a model of restricted communications, incentive compatibility is costly.

One major limitation of the present results should be mentioned:
The present results are disconnected in that the classification theorem for dominant strategy implementable mechanism is proved for the complete domain of single-peaked preferences, whereas most other results are related to the quadratic voting model. We gave a

definition of the domain specification of regular, single-crossing and tops-connected preferences and posed a conjecture that would strengthen our result. This conjecture remains open and is necessary to complement the present results.

We mention three extensions of our work that could yield valuable insights:

BIC The present thesis studied dominant strategy implementable, anonymous, non-wasteful on the one hand side and first-best mechanisms on the other side. Although dominant strategy implementable mechanisms are desirable due to their robustness, BIC is often a preferred model as it allows for more flexible mechanisms. Recall that a mechanism is Bayesian implementable if all players send messages that maximise their expected payoffs. welfare maximising, Bayesian incentive compatible mechanisms are likely to have a growing range in the size of the society—due to the intractability of Bayesian incentive compatible voting this remains an open problem. Evidence pointing in this direction is that any mechanism $g\colon \mathcal{M}^n \to \mathcal{K}$ induces a symmetric Bayesian equilibrium under mild continuity conditions on utilities and non-atomic type distributions (this is a game of finite actions and continuous types, via the existence result from Mas-Colell 1984, Theorem 2 and via purification, Radner and Rosenthal 1982, Theorem 1) and that computations for examples in the quadratic voting model for two messages show that the (injective) first-best mechanism is BIC.[11]

Proposition. *Let F be the uniform distribution on $[0,1]$ and $k=2$. Then the first-best mechanism is symmetric BIC .*

Such a flexibility might be another justification of the existence of political mandates, i.e. the political agenda of the winning party depending on the outcome of elections as studied in McMurray 2017.

Higher-dimensional Domains We study the first-best domains. Along the lines of the proof of Theorem 1, a formulation for multidimensional types and optimal vector quantisation should not be too hard to prove. As a corresponding theory for the multidimensional incentive compatible case is harder to formulate due to the existence of several generalisation of single-crossing domains to higher-dimensional types, we leave this for further research.

[11]The proof of the following result(s) can be found in the appendix.

Optimality Conditions for Representative Points We proved that both First- and Second-Best mechanisms induce an embedding of the message set $\mathcal{M}$ into the choice set $\mathcal{K}$. For first-best mechanisms, tractable necessary conditions for the choice of representative points exist. It remains a challenge to analyse the optimal position of representative points in the DIC case.

7 Proofs

Proofs for Section 3

Theorem 1 (Classification of first-best mechanisms)**.** *Let F be a square-integrable,* $[0,1]$*-valued distribution with optimal $\mathcal{M}$-quantisation $(f^{\uparrow}, f^{\downarrow})$ and variance σ^2. Then there is a first-best mechanism $g_{\text{first-best}}$ together with implementing strategies s_1, s_2, $\ldots$, s_n such that*

$$g_{\text{first-best}}(m_1, m_2, \ldots, m_n) = \frac{1}{n}\sum_{i=1}^{n} f^{\uparrow}(m_i)$$
$$s_i(x) = f^{\downarrow}(x).$$

In addition, $-W(F, g_{\text{first-best}}) = \frac{1}{n}\operatorname{MSE}^*(F) + \frac{n-1}{n}\sigma^2$.

We would like to stress that in the following proof, the quadratic specification is heavily used two conclude the following:

(a) For fixed strategies, the optimal mechanism will minimise L_2-distance of a random variable subject to measurability constraints and is hence a conditional expectation.

(b) Conditional expectations are linear.

In the interpretation of this result, one should hence keep in mind that likely even for small variation in the utility specification, the welfare maximising mechanism might be much more complex.

Proof of Theorem 1. First-best mechanisms solve the optimisation problem

$$W(F, g_{\text{first-best}}) = \frac{1}{n} \max_{\substack{s_i\colon \mathcal{T}\to\mathcal{M} \\ i=1,2,\ldots,n \\ g\colon \mathcal{M}^n\to\mathcal{K}}} \mathbb{E}\left[-\|g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) - t_i\|_2^2\right]. \tag{8}$$

For ease of notation, we will also write Z for $g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n))$. For the following formulations, we need further notation: $\mathbb{1}$ is the all-one vector, $\mathbf{t} \coloneqq (t_1, t_2, \ldots, t_n)$ and $\langle \bullet, \bullet \rangle$ denotes the Euclidean scalar product. Then the maximand can be written as the expectation of

$$-\langle Z\mathbb{1} - \mathbf{t}, Z\mathbb{1} - \mathbf{t} \rangle . \tag{9}$$

Let $\bar{t} = \frac{1}{n} \sum_{i=1}^{n} t_i$ be the average of the types of the agents. Then we can decompose (9) linearly:

$$\begin{aligned} -\langle Z\mathbb{1} - \mathbf{t}, Z\mathbb{1} - \mathbf{t} \rangle &= -\left\langle \bar{t}\mathbb{1} - \mathbf{t}, \bar{t}\mathbb{1} - \mathbf{t} \right\rangle - \left\langle (Z - \bar{t})\mathbb{1}, (Z - \bar{t})\mathbb{1} \right\rangle \\ &= -\|\mathbf{t} - \bar{t}\mathbb{1}\|_2^2 - \sum_{i=1}^{n} (g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) - \bar{t})^2. \end{aligned}$$

Here, the first equality follows by virtue of the vector orthogonality relation $\mathbf{t} - \bar{t}\mathbb{1} \perp c\mathbb{1}$ for any $c \in \mathbb{R}$, in particular $c = g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) - \bar{t}$. Re-applying expectation and the maximum, and multiplying by n,

$$nW(F, g_{\text{first-best}}) = -\mathbb{E}\left[\|\mathbf{t} - \bar{t}\mathbb{1}\|_2^2\right] - \max_{\substack{s_i \colon \mathcal{T} \to \mathcal{M} \\ i=1,2,\ldots,n \\ g \colon \mathcal{M}^n \to \mathcal{K}}} \mathbb{E}\left[\sum_{i=1}^{n} (g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) - \bar{t})^2\right]. \tag{10}$$

To find the maximisers $s_1, s_2, \ldots, s_n$ and $g_{\text{first-best}}$, we can neglect the first summand that is constant and consider a minimisation problem instead of maximising the negative. Therefore, the following minimisation problem has the same maximisers as (8):

$$\begin{aligned} &\frac{1}{n} \min_{\substack{s_i \colon \mathcal{T} \to \mathcal{M} \\ i=1,2,\ldots,n \\ g \colon \mathcal{M}^n \to \mathcal{K}}} \mathbb{E}\left[\sum_{i=1}^{n} (g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) - \bar{t})^2\right] \\ &\qquad = \min_{\substack{s_i \colon \mathcal{T} \to \mathcal{M} \\ i=1,2,\ldots,n \\ g \colon \mathcal{M}^n \to \mathcal{K}}} \mathbb{E}\left[(g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) - \bar{t})^2\right] \end{aligned} \tag{11}$$

Consider now the randm variable $Z(t_1, t_2, \ldots, t_n)$. The property that Z can only depend on t_i through s_i and is deterministic given the $s_1(t_1), s_2(t_2), \ldots, s_n(t_n)$ can be reformulated in terms of measurability. It is equivalent to Z being measurable with respect to the σ-algebra generated by events $\{\{s_i(t_i) = m\}\}_{\substack{i=1,2,\ldots,n \\ m \in \mathcal{M}}}$. Hence, we can rewrite the optimisation problem as

$$\min_{\substack{s_i \colon \mathcal{T} \to \mathcal{M} \\ i=1,2,\ldots,n}} \left(\min_{Z \ \{\{s_i(t_i) = m\}\}_{\substack{i=1,2,\ldots,n \\ m \in \mathcal{M}}}\text{-mb.}} \mathbb{E}[(Z - \bar{t})^2] \right) .$$

Furthermore, there the mapping

$$\{\text{functions } \mathcal{T} \to \mathcal{M}\}^n \to \{|\mathcal{M}|\text{-partitions of } \mathcal{T}\}^n,$$
$$(s_i \colon \mathcal{T} \to \mathcal{M})_{i=1,2,\dots,n} \mapsto (\{s_j^{-1}(\{m\})|\ m \in \mathcal{M}\})_{j=1,2,\dots,n} \quad (12)$$

from tuples of strategies to tuples of partitions is bijective. Hence, we can re-parametrise the optimisation problem as

$$\min_{\substack{\{A_{im}|\ m\in\mathcal{M}\} \text{ partition of } \mathcal{T}\\ i=1,2,\dots,n}} \left(\min_{\substack{Z\ \{\{t_i \in A_{im}\}\}_{i=1,2,\dots,n}\text{-mb.}\\ m\in\mathcal{M}}} \mathbb{E}[(Z-\bar{t})^2] \right).$$

Fix optimal partitions $\{A_{im}|\ m \in \mathcal{M}\}$, $i = 1, 2, \dots, n$ and consider the inner minimisation problem. From probability theory we know that the conditional expectation is the L^2-projection of a random variable onto the subspace of measurable random variables with respect to the σ-algebra it is conditioned on, Billingsley 2008, Theorem 34.16.[12] Hence,

$$Z = \mathbb{E}[\bar{t}|\sigma(\{\{t_i \in A_{im}\}\}_{\substack{i=1,2,\dots,n\\ m\in\mathcal{M}}})]$$

Then by linearity of the conditional expectation, Billingsley 2008, Theorem 34.2 (ii) and by the independence of t_i from $t_{i'}$, $i \neq i'$ for still fixed optimal partitions $\{A_{im}|\ m \in \mathcal{M}\}$, $i = 1, 2, \dots, n$,

$$Z = \frac{1}{n}\sum_{i'=1}^{n} \mathbb{E}[t_{i'}|\ \sigma(\{\{t_i \in A_{im}\}\}_{\substack{i=1,2,\dots,n\\ m\in\mathcal{M}}})] = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[t_i|\ \sigma(\{\{t_i \in A_{im}\}\}_{m\in\mathcal{M}})]. \quad (13)$$

Substituting (13) into (11), we obtain

$$\begin{aligned}
&\min_{\substack{s_i\colon \mathcal{K}\to\mathcal{M}\\ i=1,2,\dots,n\\ g\colon \mathcal{M}^n\to\mathcal{K}}} \mathbb{E}\left[(g(s_1(t_1), s_2(t_2), \dots, s_n(t_n)) - \bar{t})^2\right] \\
&= \min_{\substack{\{A_{im}|\ m\in\mathcal{M}\} \text{ partition of } \mathcal{T}\\ i=1,2,\dots,n}} \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[t_i|\ \sigma(\{\{t_i \in A_{im}\}\}_{m\in\mathcal{M}})] - \frac{1}{n}\sum_{i=1}^{n} t_i\right)^2\right] \\
&= \min_{\substack{\{A_{im}|\ m\in\mathcal{M}\} \text{ partition of } \mathcal{T}\\ i=1,2,\dots,n}} \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n}(\mathbb{E}[t_i|\ \sigma(\{\{t_i \in A_{im}\}\}_{m\in\mathcal{M}})] - t_i)\right)^2\right] \\
&= \frac{1}{n^2}\min_{\substack{\{A_{im}|\ m\in\mathcal{M}\} \text{ partition of } \mathcal{T}\\ i=1,2,\dots,n}} \mathbb{E}\left[\sum_{i=1}^{n}(\mathbb{E}[t_i|\ \sigma(\{\{t_i \in A_{im}\}\}_{m\in\mathcal{M}})] - t_i)^2\right]
\end{aligned}$$

[12] Because both $\mathcal{K}$ and $\mathcal{T}$ are bounded and F is a probability measure, L^2 integrability of Z is given.

$$= \frac{1}{n^2}\sum_{i=1}^{n} \min_{\substack{\{A_{im}|\ m\in\mathcal{M}\}\\ \text{partition of } \mathcal{T}}} \mathbb{E}\left[(\mathbb{E}[t_i|\ \sigma(\{\{t_i \in A_{im}\}\}_{m\in\mathcal{M}})] - t_i)^2\right]$$

The second to last equality holds as $\mathbb{E}[t_i|\ \sigma(\{\{t_i \in A_{ij}\}\}_{j\in[k]})] - t_i$ are measurable with respect to t_i, zero-mean, and the t_i are independent. In particular, the $\mathbb{E}[t_i|\ \sigma(\{\{t_i \in A_{ij}\}\}_{j\in[k]})] - t_i$ are uncorrelated zero mean, compare Billingsley 2008, Theorem 4.2. The last inequality holds as the minimisation in the second to last line is performed for all $i = 1, 2, \dots, n$ separately.

Finally, as the t_i are identically distributed, we can select a common maximising partition for each summand, i.e. the partitions can be chosen as $A_m \coloneqq A_{im} = A_{i'm}$, $1 \le i, i' \le n, m \in \mathcal{M}$. As the t_i are identically distributed, we neglect the subscript and write t instead of t_i. Hence

$$\begin{aligned}&\frac{1}{n^2}\sum_{i=1}^{n} \min_{\substack{\{A_{im}|\ m\in\mathcal{M}\}\\ \text{partition of } \mathcal{T}}} \mathbb{E}\left[(\mathbb{E}[t_i|\ \sigma(\{\{t_i \in A_{im}\}\}_{m\in\mathcal{M}})] - t_i)^2\right]\\ &= \frac{1}{n} \min_{\substack{\{A_m|m\in\mathcal{M}\}\\ \text{partition of } \mathcal{T}}} \mathbb{E}[(\mathbb{E}[t|\ \sigma(\{\{t \in A_m\}\}_{m\in\mathcal{M}})] - t)^2]\end{aligned}$$

Using (12) to re-parametrise once more,

$$\frac{1}{n} \min_{\substack{\{A_m|m\in\mathcal{M}\}\\ \text{partition of } \mathcal{T}}} \mathbb{E}[(t - \mathbb{E}[t|\ \sigma(\{\{t \in A_m\}\}_{m\in\mathcal{M}})])^2] = \frac{1}{n}\min_{s\colon \mathcal{T}\to\mathcal{M}} \mathbb{E}[(t - \mathbb{E}[t|\ t \in s^{-1}(\{s(t)\})])^2].$$

By (5) and the fact that s is chosen to minimise L^2-distance, $\mathbb{E}[t|\ t \in s^{-1}(\{s(t)\})] = f^{\uparrow}(s(t))$. Again by the fact that s is chosen to minimise L^2-distance and (4), $s = f^{\downarrow}$. Finally, upon substituting,

$$W(F, g_{\text{first-best}}) = \mathbb{E}\left[\|\mathbf{t} - \bar{t}\mathbb{1}\|_2^2\right] + \min_{\substack{f^{\downarrow}\colon \mathcal{T}\to\mathcal{M}\\ f^{\uparrow}\colon \mathcal{M}\to\mathcal{T}}} \mathbb{E}[(X - f^{\uparrow}(f^{\downarrow}(X)))^2] = \frac{n-1}{n}\sigma^2 + \frac{1}{n}\,\mathrm{MSE}^*(F),$$

which yields the claim upon rearranging. □

Proposition 2 (Convergence of average welfare for first-best mechanism)**.** *Let F be a distribution on $[0,1]$ with variance σ^2. For any $\mathcal{M}$ with $|\mathcal{M}| = k$, we have*

$$\frac{1}{12nk^2} \le -\frac{n-1}{n}\sigma^2 - W(F, g_{\text{first-best}}) \le \frac{1}{4nk^2}, \tag{6}$$

hence $\frac{n-1}{n}\sigma^2 + W(F, g_{\text{first-best}}) \in \Theta(k^{-2}) \cap \Theta(n^{-1})$.

The following proof is an adaption of Bergemann, Shen, Xu, and E. M. Yeh 2011, Proposition 3, and up to a constant of $\frac{1}{2}$, the computations are identical. As Bergemann, Shen, Xu, and E. M. Yeh 2011 studies mechanism design with monetary transfers in a linear quadratic model, we give the proof of the upper bound in detail and refer to Bergemann, Shen, Xu, and E. M. Yeh 2011, where the lower bound is provided in sufficient detail.

Proof of Proposition 2. By Theorem 1, $-\frac{n-1}{n}\sigma - W(F, g_{\text{first-best}}) = \frac{1}{n}\,\text{MSE}^*(F)$. Hence, it suffices to prove $\frac{1}{12k^2} \leq \text{MSE}^*(F) \leq \frac{1}{4k^2}$.

First note that by the theorem of total expectation for any quantisation $(f^{\downarrow}, f^{\uparrow})$

$$\begin{aligned}\text{MSE}(F, (f^{\downarrow}, f^{\uparrow})) &= \mathbb{E}[(X - f^{\uparrow}(f^{\downarrow}(X)))^2] \\ &= \sum_{m \in \mathcal{M}} \underbrace{\mathbb{E}[(t - f^{\uparrow}(m_i))^2 | f^{\uparrow}(t) = m]}_{\text{Var}[t|f^{\downarrow}(t)=m]} \mathbb{P}[f^{\uparrow}(t) = m]\end{aligned}$$

Note that $\text{MSE}^*(F) \leq \text{MSE}(F, (f^{\downarrow}, f^{\uparrow}))$. Choose an enumeraion $\{m_i\}_{i=1}^k$ of $\mathcal{M}$. Then define

$$f^{\downarrow}(x) = m_i \text{ if } \frac{i-1}{k} \leq x \leq \frac{i}{k} \qquad\qquad f^{\uparrow}(m_i) = \mathbb{E}[t|\ f^{\uparrow}(t) = m_i].$$

We remark that, conditional on $f^{\uparrow}(t) = m_i$,

$$\frac{i-1}{k} \leq t \leq \frac{i}{k}.$$

Hence, conditionally, the range of t has length $\frac{1}{k}$. By the inequality $\text{Var}[t|f^{\downarrow}(t) = m] \leq (\frac{1}{2k})^2 = \frac{1}{4k^2}$,

$$\begin{aligned}\text{MSE}^*(F) &\leq \text{MSE}(F, (f^{\downarrow}, f^{\uparrow})) \\ &= \sum_{i=1}^{k} \text{Var}[t|\ f^{\uparrow}(t) = m_i]\mathbb{P}[f^{\downarrow}(t) = m_i] \\ &\leq \sum_{i=1}^{k} \mathbb{P}[f^{\uparrow}(t) = m_i]\frac{1}{4k^2} = \frac{1}{4k^2}.\end{aligned}$$

The lower bound is attained for the uniform distribution, cf. Bergemann, Shen, Xu, and E. M. Yeh 2011, (5) and Example 1. □

Proofs for Section 4

Theorem 4 (Classification of dominant incentive compatible anonymous non-wasteful mechanisms, single-peaked)**.** *Let $\mathcal{T}$ be the set of single-peaked preferences on $\mathcal{K}$. Then $g\colon \mathcal{T}^n \to \mathcal{K}$ is anonymous, non-wasteful and DIC implementable by surjective strategies if and only if it is an embedded generalised median voting rule.*

Proof of Theorem 4. This follows from Lemma 2 and Theorem 9. □

First, we start with some notation. For $A \subseteq \mathcal{K}$ dente by $\mathcal{T}_A \coloneqq \{t|_{A\times A} \mid t \in \mathcal{T}\}$ the preferences on $\mathcal{K}$ restricted to A with the projection function $p\colon \mathcal{T} \twoheadrightarrow \mathcal{T}_A, t \mapsto t|_{A\times A}$.

Second, we need a further definition of a mechanism: A social choice function $f\colon \mathcal{T}^n \to \mathcal{K}$ is called $(n+1)$*-parameter generalised median voting rule* if there are $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{K}$, the *phantom ballots* such that

$$f(t_1, t_2, \ldots, t_n) = \operatorname{med}\{\tau_{\mathcal{K}}(t_1), \tau_{\mathcal{K}}(t_2), \ldots, \tau_{\mathcal{K}}(t_n), \alpha_1, \alpha_2, \ldots, \alpha_{n+1}\}.$$

The difference to generalised median voting schemes is that there are $n+1$ instead of $n-1$ phantom ballots.

One easily sees (compare e.g. Weymark 2011, p. 548 bottom) that if we require such a social choice function to be surjective, then it is a generalised median voting rule.

Lemma 1. *An $(n+1)$-parameter generalised median voting rule $f\colon \mathcal{T}^n \to \mathcal{K}$ is surjective if and only if is a generalised median voting rule.*

Proof. By definition of $(n+1)$-parameter generalised median voting rules, f is a generalised median voting rule if there are two phantom ballots $\alpha_i = \underline{k}$, $\alpha_{i'} = \overline{k}$ where f's phantom ballots are the other $n-1$ phantom ballots. It hence suffices to show that there must be two such phantom ballots. Assume that the generalised median voting rule is surjective and that the largest resp. smallest phantom ballot is not on $\overline{k}$ resp. $\underline{k}$. Then there are $n+1$ phantom ballots on values smaller than $\overline{k}$ resp. larger than $\underline{k}$. But then no matter how the voter ballots are placed, the median of the $n+1$ phantom ballots and n voter ballots will be smaller than $\overline{k}$ resp. larger than $\underline{k}$. Hence the rule cannot be surjective. By contradiction, we obtain the result. □

Third, we define two properties of preference domains:

Definition (Properties A and B)**.** *Let $\mathcal{T}$ be a preference domain. It is said to have* property A *if for any subset $\mathcal{T}$ of A, $\tau_A\colon \mathcal{T}_A \to A$ is monotone and surjective.*[13]

It is said to have property B *if for any finite $A \subseteq \mathcal{K}$ a social choice function $\mathcal{T}_A^n \to A$ is strategy-proof, surjective and anonymous if and only if it is an generalised median voting rule on $\mathcal{T}_A$.*

Lemma 2. *The preference domain of single-peaked preferences satisfies properties A and B.*

Proof. We start with property A. Monotonicity follows by definition of the order on $\mathcal{T}$. To prove surjectivity, we define the strict partial order order $\tilde{\preceq}$ on $\mathcal{K}$ by

$$k > x > y \vee k < x < y \Rightarrow x \succeq y.$$

We can refine this partial order to a linear order by Szpilrajn 1930, Theorem on p. 386 (assuming the axiom of choice). This order is by definition single-crossing.

Let us turn to property B. We show that $\mathcal{T}_A$ is the set of single-peaked preferences on A. Then Weymark 2011, Theorem 4 in connection with Lemma 1 is sufficient to establish the claim.

Let first $t_A \in \mathcal{T}_A$ and $t_A = p(t)$. By single-peakedness, $\tau_A(t_A) = \min\{k \in A| \ k \geq \tau_{\mathcal{K}}(t)\}$ or $\tau_A(t_A) = \max\{k \in A| \ k < \tau_{\mathcal{K}}(t)\}$. We assume the further, the latter case is similar. By single-peakedness, for any $k \in A$ such that $k > \tau_A(t)$ we have $k \geq \tau_{\mathcal{K}}(t)$ and hence $k \prec^{t_A} \tau_A(t_A)$. Furthermore, for any other k, is must be that $k \leq \max\{t \in A| \ t < \tau_{\mathcal{K}}(t)\} \preceq^{t_A} \tau_A(t_A)$, which implies by single-crossing that $k \preceq \tau_{\mathcal{K}}(t)$. Hence, every preference relation in $\mathcal{T}_A$ is single-crossing. Conversely, let t_A be a single-crossing preference on A. We would like to show that there is $t \in \mathcal{T}$ such that $p(t) = t_A$. Define such a t by the completion of the strict partial order $\preceq$ that is defined by

$$(\tau_A(t_A) > x > y \vee \tau_A(t_A) < x < y) \Rightarrow x \preceq y,$$

which exists again by Szpilrajn 1930, Theorem on p. 386. □

Then the following theorem implies Theorem 4.

Theorem 9 (Classification of dominant incentive compatible anonymous non-wasteful mechanisms, axiomatic)**.** *Let $\mathcal{T}$ be a preference domain that satisfies properties A and*

[13] Note that to say that τ_A is surjective is to say that $\mathcal{T}_A$ is a regular domain.

B. Then $g\colon \mathcal{T}^n \to \mathcal{K}$ is anonymous, non-wasteful and DIC implementable by surjective strategies if and only if it is an embedded generalised median voting rule.

Before coming to the proof of Theorem 9, we start with a few more lemmas.[14]

Lemma 3. *If $g : \mathcal{M}^n \to \mathcal{K}$ is anonymous, non-wasteful and dominant strategy implementable by strategies $s_1, s_2, \ldots, s_n\colon \mathcal{T} \to \mathcal{M}$, then $s_i = s_{i'}, 1 \leq i, i' \leq n$.*

Lemma 4. *Let $A \subseteq \mathcal{K}$. Let $x, y \in A$ be adjacent, i.e. there is no $z \in A$ such that $x < z < y$. Let $\alpha_1, \ldots, \alpha_{n-1} \in [\underline{k}, x] \cup [y, \overline{k}]$ and $i = 1, 2, \ldots, n$. Then for $h\colon A \to A, (a_1, a_2 \ldots, a_n) \mapsto \operatorname{med}\{a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ there is $a_{-i} \in \mathcal{A}^{n-1}$ such that*

$$x = h(x, a_{-i}) \neq h(y, a_{-i}) = y.$$

Proof of Theorem 9. We first show that every embedded generalised median voting rule g is anonymous, non-wasteful and dominant strategy implementable by surjective strategies. Anonymity is clear and and non-wastefulness follows by Lemma 4. Consider strategies

$$s_i(t) = s_j(t) = \iota^{-1}(\tau_{\operatorname{range} g}(t)).$$

These strategies are well-defined and surjective as ι is bijective (this follows from α_1, α_2, $\ldots$, $\alpha_{n-1} \in \operatorname{range} \iota$) and by the surjectivity part of property A.

Concerning dominant strategy implementability, fix $m_{-i} \in \mathcal{M}^{n-1}$. Then again by the surjectivity part of property A and bijectivity of ι, there is t_{-i} such that $m_{i'} = \iota^{-1}(\tau_{\operatorname{range} g}(t_{i'}))$, $i' = 1, 2, \ldots, n$. As an abuse of notation, we in the following equation $f(t_{-i})$ instead of $(f(t_1), f(t_2), \ldots, f(t_n))$ for several functions f. By definition,

$$\begin{aligned} g(s_1(t_1), m_{-i}) &= \operatorname{med}(\iota(\iota^{-1}(\tau_{\operatorname{range} g}(t_i))), \iota_{-i}(m_{-i}), \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \\ &= \operatorname{med}(\iota(\iota^{-1}(\tau_{\operatorname{range} g}(t_1))), \iota_{-i}(\iota_{-i}^{-1}(\tau_{\operatorname{range} g}(t_{-i}))), \alpha_1, \alpha_2, \ldots, \alpha_{n-1}) \\ &= \operatorname{med}(\tau_{\operatorname{range} g}(t_i), \tau_{\operatorname{range} g}(t_{-i}), \alpha_1, \alpha_2, \ldots, \alpha_{n-1}). \end{aligned}$$

The social choice function

$$f'\colon \mathcal{T}^n_{\operatorname{range} g} \to \mathcal{T}_{\operatorname{range} g}, g(t_1, t_2, \ldots, t_n) = \operatorname{med}(\tau_{\operatorname{range} g}(t_i), \tau_{\operatorname{range} g}(t_{-i}), \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$$

is a generalised median voting rule, and hence strategy-proof by property B. Hence

$$g(s_1(t_i), m_{-i}) = \operatorname{med}(\tau_{\operatorname{range} g}(t_i), \tau_{\operatorname{range} g}(t_{-i}), \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$$

[14]The proof of the following result(s) can be found in the appendix.

$$\succeq^{t_i} \operatorname{med}(\tau_{\operatorname{range} g}(t_i'), \tau_{\operatorname{range} g}(t_{-i}), \alpha_1, \alpha_2, \ldots, \alpha_{n-1})$$
$$= g(s_i(t_i'), m_{-i}),$$

which implies dominant strategy implementability. We now prove the contrary.

(a) Fix a mechanism $g\colon \mathcal{M}^n \to \mathcal{K}$ that is dominant strategy implementable by s_1, s_2, …, $s_n\colon \mathcal{T} \to \mathcal{M}$. The social choice function

$$f\colon \mathcal{T}^n \to \mathcal{K}, f(t_1, t_2, \ldots, t_n) = g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n))$$

is strategyproof. Indeed, fix any $i = 1, 2, \ldots, n$, $t_{-i} \in \mathcal{T}^{n-1}$ and $t_i, t' \in \mathcal{T}$. Then

$$f(t_i, t_{-i}) = f(s_i(t_i), s_{-i}(t_{-i})) \succeq^{t_i} g(s_i(t'), s_{-i}(t_i)) = f(t', s_{-i}(t_{-i})),$$

where the preference is a consequence of $s_1, s_2, \ldots, s_n$ forming a dominant strategy equilibrium.

(b) Let $\mathcal{F} = \operatorname{range}(f)$. f does only depend on the preferences on the set $\mathcal{F}$, i.e. for any $t_i, t_i' \in \mathcal{T}$, $t_{-i} \in \mathcal{T}^{n-1}$ with $p(t_i) = p(t_i')$ it holds that $f(t_i, t_{-i}) = f(t_i', t_{-i})$. Indeed, $f(t_i, t_{-i}) \succeq^{t_i} f(t_i', t_i)$ and $f(t_i, t_{-i}) \preceq^{t_i'} f(t_i', t_{-i})$ by strategy-proofness but also $t(t_i, t_{-i}) \preceq^{t_i} f(t_i', t_{-i})$ as $p(t_i) = p(t_i')$. Hence, the assertion follows by antisymmetry.

We can conclude that the function $\widetilde{f}\colon \mathcal{T}_{\mathcal{F}}^n \to \mathcal{K}$ that satisfies $\widetilde{f}(p(t_1), p(t_2), p(t_n)) = f(t_1, t_2, \ldots, t_n)$ is well-defined. By definition of f, we also have

$$\tilde{f}(p(t_1), p(t_2), \ldots, p(t_n)) = g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) \tag{14}$$

Note in particular, that as f is strategy-proof, so must be $\tilde{f}$.

(c) $\tilde{f}$ is anonymous. Indeed, Let $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n \in \mathcal{T}_{\mathcal{F}}$ and $\pi \in \mathcal{S}_n$. Then, as p is surjective, there are $t_1, t_2, \ldots, t_n \in \mathcal{T}$ such that $p(t_i) = \tilde{t}_i$ for all $i = 1, 2, \ldots, n$. Hence,

$$\begin{aligned}
\tilde{f}(\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_n)) &= \tilde{f}(p(t_1), p(t_2), \ldots, p_(t_n)) \\
&= g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) \\
&= g(s_{\pi 1}(t_{\pi 1}), s_{\pi(2)}(t_{\pi(2)}), \ldots, s_{\pi(n)}(t_{\pi(n)})) \\
&= \tilde{f}(p(t_{\pi(1)}), p(t_{\pi(2)}), \ldots, p_(t_{\pi(n)})) \\
&= \tilde{f}(\tilde{t}_{\pi(1)}, \tilde{t}_{\pi(2)}, \ldots, \tilde{t}_{\pi(n)})).
\end{aligned}$$

Hence, $\tilde{f}$ is anonymous and strategy-proof on the preference domain $\mathcal{T}_{\mathcal{F}}$.

(d) By property B $\tilde{f}$ must be an $(n+1)$-parameter generalised median voting rule as it is strategyproof and anonymous. As it is surjective, it must be a generalised median voting rule by Lemma 1. In particular, it is tops-only, i.e. depends on types t_i only through $\tau_{\mathcal{K}}(t_i)$. Hence, there is a function $f' \colon \mathcal{F}^n \to \mathcal{K}$ such that

$$\begin{aligned}\tilde{f}(p(t_1), p(t_2), \ldots, p(t_n)) &= f'(\tau_{\mathcal{F}}(p(t_1)), \tau_{\mathcal{F}}(p(t_2)), \ldots, \tau_{\mathcal{F}}(p(t_n))) \\ &= f'(\tau_{\mathcal{F}}(t_1), \tau_{\mathcal{F}}(t_2), \ldots, \tau_{\mathcal{F}}(t_n)),\end{aligned} \tag{15}$$

where $\tau_{\mathcal{F}}$ denotes the top alternative within the set $\mathcal{F}$. Note that in a slight abuse of notation, $\tau_{\mathcal{F}} \colon \mathcal{T}_{\mathcal{F}} \to \mathcal{F}$ in the second to last term and $\tau_{\mathcal{F}} \colon \mathcal{T} \to \mathcal{F}$ in the last term are denoted by the same symbol. Hence,

$$f'(\tau_{\mathcal{F}}(t_1), \tau_{\mathcal{F}}(t_2), \ldots, \tau_{\mathcal{F}}(t_n)) = g(s_1(t_1), s_2(t_2), \ldots, s_n(t_n)) = f(t_1, t_2, \ldots, t_n). \tag{16}$$

(e) Consider the equivalence relation $\sim$ on $\mathcal{T}$ defined by

$$t \sim t' \colon \iff f(t, t_{-i}) = f(t', t_{-i}), \text{ for any } t_{-i} \in \mathcal{T}$$

Then equivalence classes of this equivalence relation are connected sets in the ordered set $\mathcal{T}$. Indeed, assume the contrary. Then there are $t' < t < t'' \in \mathcal{T}$ such that $t' \sim t''$, $t' \not\sim t$ and $t \not\sim t''$. By the monotonicity part of property A, we know that $\tau_{\mathcal{F}}(t') \leq \tau_{\mathcal{F}}(t) \leq \tau_{\mathcal{F}}(t'')$. Furthermore, as $t \not\sim t'$ there is $t_{-i} \in \mathcal{T}^{n-1}$ such that $f(t', t_{-i}) \neq f(t, t_{-i})$. As $f(t_i, t_{-i}) = f'(\tau_{\mathcal{F}}(t_i), \tau_{\mathcal{F}}(t_{-i}))$ and $f'(\bullet, \tau_{\mathcal{F}}(t_{-i}))$ is the marginal of a generalised median voting rule, which is monotone, it must be that

$$f(t', t_{-i}) = f'(\tau_{\mathcal{F}}(t'), \tau_{\mathcal{F}}(t_{-i})) < f'(\tau_{\mathcal{F}}(t), \tau_{\mathcal{F}}(t_{-i})) \leq f'(\tau_{\mathcal{F}}(t''), \tau_{\mathcal{F}}(t_{-i})) = f(t'', t_{-i}),$$

hence $f(t', t_{-i}) < f(t'', t_{-i})$ and hence $t' \not\sim t''$, which is a contradiction. Hence, the equivalence classes are connected sets in $\mathcal{T}$. As $s_i = s_j$, $1 \leq i, j \leq n$ by Lemma 3, we can from now on suppress the subscript and write just s for the strategies.

(f) Let

$$\mathcal{A} := \{s^{-1}(\{j\}) | j \in \mathcal{M}\} \qquad\qquad \mathcal{B} := \{\tau_{\mathcal{F}}^{-1}(\{f\}) |\, f \in \mathcal{F}\}.$$

We show that $\mathcal{A} = \mathcal{B}$. This implies that there is a bijection $\iota \colon \mathcal{M} \to \mathcal{F} \hookrightarrow \mathcal{K}$ such that $\iota(s(t)) = \tau_{\mathcal{F}}(t)$ for any $t \in \mathcal{T}$. Indeed, one can define a function ι function by

$$\iota(j) = f \iff s^{-1}(\{j\}) \subseteq \tau_{\mathcal{F}}^{-1}(\{f\})$$

The function is well-defined as $s^{-1}(\{j\}) \subseteq \tau_{\mathcal{F}}^{-1}(\{f\}), \tau_{\mathcal{F}}^{-1}(\{f'\})$ implies by $\mathcal{A} = \mathcal{B}$ that $\tau_{\mathcal{F}}^{-1}(\{f\}) = \tau_{\mathcal{F}}^{-1}(\{f'\})$ which implies $f = f'$. In addition, by definition, $f = \iota(s(t))$ has the property that $t \in s^{-1}(\{s(t)\}) \subseteq \tau_{\mathcal{F}}^{-1}(\{f\})$, hence $\tau_{\mathcal{F}}(t) = f = \iota(s(t))$. So this function satisfies $\tau_{\mathcal{F}}(t) = \iota(s(t))$. Furthermore, the function is injective. Indeed, if $\iota(j) = \iota(j')$, then $s_i^{-1}(\{j\}), s_i^{-1}(\{j'\}) \subseteq \tau_{\mathcal{F}}^{-1}(\{f\})$ which implies $s_i^{-1}(\{j\}) = s_i^{-1}(\{j'\})$ by $\mathcal{A} = \mathcal{B}$ which in turn implies $j = j'$.

(g) Observe that $\bigcup_{A\in\mathcal{A}} A = \bigcup_{B\in\mathcal{B}} = B$ are disjoint unions by definition. Hence $\mathcal{A} = \mathcal{B}$ follows upon proving the following two claims:

   (a) For any $A \in \mathcal{A}$ resp. $B \in \mathcal{B}$ and $t_A, t'_A \in A$, $t_B, t'_B \in B$ it holds that $t_A \sim t'_A$ resp. $t_B \sim t'_B$.

   (b) If $A, A' \in \mathcal{A}$ resp. $B, B' \in \mathcal{B}$ and there are $t_A \in A$, $t_{A'} \in A'$ resp. $t_B \in B$, $t_{B'} \in B'$ such that $t_A \sim t_{A'}$ resp. $t_B \sim t_{B'}$, then $A = A'$, $B = B'$.

These two claims imply that $\mathcal{A}$ and $\mathcal{B}$ are both the collection of equivalence classes of $\sim$ and hence $\mathcal{A} = \mathcal{B}$.

We first prove item (a). Let $s(t_A) = s(t'_A)$. Choose any $t_{-i} \in \mathcal{T}^{n-1}$. Then

$$\begin{aligned} f(t_A, t_{-i}) &= g(s_i(t_A), s_{-i}(t_{-i})) \\ &= g(s_i(t'_A), s_{-i}(t_{-i})) \\ &= f(t'_A, t_{-i}), \end{aligned}$$

As $t_{-i} \in \mathcal{T}^{n-1}$ was arbitrary, $t_A \sim t'_A$. Now assume that $\tau_{\mathcal{F}}(t_B) = \tau_{\mathcal{F}}(t'_B)$. Then let $t_{-i} \in \mathcal{T}^{n-1}$. It follows that

$$\begin{aligned} f(t_B, t_{-i}) &= g(s_i(t_B), s_{-i}(t_{-i})) \\ &= f'(\tau_{\mathcal{F}}(t_B), \tau_{\mathcal{F}}(t_{-i})) \\ &= f'(\tau_{\mathcal{F}}(t'_B), \tau_{\mathcal{F}}(t_{-i})) \\ &= g(s_i(t'_B), s_{-i}(t_{-i})) \\ &= f(t'_B, t_{-i}). \end{aligned}$$

As $t_{-i} \in \mathcal{T}^{n-1}$ was arbitrary, $t_B \sim t'_B$. This proves item (a).

We proceed with item (b). Consider first $\mathcal{A}$. Let $A, A' \in \mathcal{A}$, $t_A \in A, t_{A'} \in A'$, $t_A \sim t_{A'}$. Let $s(t_A) = m$ and $s(t_{A'}) = m'$. Then

$$g(m, s_{-i}(t_{-i})) = f(t_A, t_{-i}) = f(t_{A'}, t_{-i}) = g(m', s_{-i}(t_{-i}))$$

By surjectivity of $s\colon \mathcal{T} \to \mathcal{M}$, it holds for any $m_{-i} \in \mathcal{M}^{n-1}$,

$$g(m, m_{-i}) = g(m', m_{-i}),$$

Hence, by non-wastefulness $m = m'$, which implies $A = A'$.

Finally, consider $\mathcal{B}$. Let $\tilde{t}_{\tilde{B}}, \tilde{t}_{\tilde{B}'} \in \mathcal{B}$, $\tilde{t}_{\tilde{B}} \sim \tilde{t}_{\tilde{B}'}$ and assume $\tilde{B} \neq \tilde{B}'$. Note in particular, that this implies by item (a) that $\tilde{B}, \tilde{B}'$ lie one equivalence class with respect to $\sim$. As the equivalence classes w.r.t $\sim$ are connected in $\mathcal{T}$ by item (e), there must be sets $B, B' \in \mathcal{B}$ and $t_B \in B$, $t_{B'} \in B'$, $t_B \sim t_{B'}$ such that $\tau_{\mathcal{F}}(t_B) = x$ and $\tau(t_{B'}) = y$ and x, y are adjacent in $\mathcal{F}$. By Lemma 4 and the surjectivity part of property A, there is t_{-i} such that

$$f(t, t_{-i}) \neq f(t', t_{-i}),$$

contradicting $t_B \sim t_{B'}$. Hence, $B = B'$.

(h) Finally, let $\iota\colon \mathcal{M} \to \mathcal{F} \subseteq \mathcal{K}$ be a bijection such that $\iota(s(t)) = \tau_{\mathcal{F}}(t)$ for any $t \in \mathcal{T}$. By surjectivity of s, there are $t_i \in \mathcal{T}$ such that $s(t_i) = m_i$. Then

$$\begin{aligned} g(m_1, m_2, \ldots, m_n) &= g(s(t_1), s(t_2), \ldots, s(t_n)) \\ &= f'(\tau_F(t_1), \tau_{\mathcal{F}}(t_2), \ldots, \tau_{\mathcal{F}}(t_n)) \\ &= f'(\iota(s(t_1)), \iota(s(t_2)), \ldots, \iota(s(t_n))) \\ &= f'(\iota(m_1), \iota(m_2), \ldots, \iota(m_n)) \end{aligned}$$

This shows that g is an embedded generalised median voting scheme.

□

Proposition 7 (Characterisation of strategyproof anonymous mechanisms for non-interval ranges)**.** *Let $\mathcal{T}$ be set of single-peaked preferences. Let $f : \mathcal{T}^n \to \mathcal{K}$ be a social choice function. Then, f is strategy-proof and anonymous if and only if there there are $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ such that for*

$$f'\colon \text{range}\, f \to \text{range}\, f, \quad f'(t_1, t_2, \ldots, t_n) = \text{med}\{t_1, t_2, \ldots, t_n, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}\} \tag{7}$$

it holds that

$$f(t_1, t_2, \ldots, t_n) = f'(\tau_{\text{range}\, f}(t_1), \tau_{\text{range}\, f}(t_2), \ldots, \tau_{\text{range}\, f}(t_n)).$$

Proof of Proposition 7. Let $f\colon \mathcal{T}^n \to \mathcal{K}$ be a strategy-proof and anomymous mechanism. We know by the proof of Theorem 9 item (b) that there is a function $\tilde{f}\colon \mathcal{T}^n_{\operatorname{range} f} \to \operatorname{range} f$ that is strategyproof and anonymous by the proof of Theorem 9 item (c) and such that

$$f(t_1, t_2, \ldots, t_n) = \tilde{f}(p(t_1), p(t_2), \ldots, p(t_n)),$$

where $p\colon \mathcal{T} \twoheadrightarrow \mathcal{T}_A$ is the projection of preference types onto the set $\mathcal{T}_A = \{t|_{A\times A} | \; t \in \mathcal{T}\}$. From the proof of Lemma 2, we know that $\mathcal{T}_{\operatorname{range} f}$ is the set of single-peaked preferences on $\operatorname{range} f$. Note in particular, as $\tilde{f}$ is surjective, and, by definition, it has an interval range in $\operatorname{range} f$. Hence, by Weymark 2011, Theorem 4, this function is an $(n+1)$-parameter generalised median voting rule. It is surjective by definition, hence a generalised median voting rule by Lemma 1. Hence, there is a function f' of the form (7) such that

$$\begin{aligned} f(t_1, t_2, \ldots, t_n) &= \tilde{f}(p(t_1), p(t_2), \ldots, p(t_n)) \\ &= f'(\tau_{\operatorname{range} f}(p(t_1)), \tau_{\operatorname{range} f}(p(t_2)), \ldots, \tau_{\operatorname{range} f}(p(t_n))) \\ &= f'(\tau_{\operatorname{range} f}(t_1), \tau_{\operatorname{range} f}(t_2), \ldots, \tau_{\operatorname{range} f}(t_n)). \end{aligned}$$

On the other hand, let $f(t_1, t_2, \ldots, t_n) = f'(\tau_{\operatorname{range} f}(t_1), \tau_{\operatorname{range} f}(t_2), \ldots, \tau_{\operatorname{range} f}(t_n))$ for a function f' of form (7). Anonymity of this rule is clear. For strategy-proofness, note that f is tops-only and hence only depends on the preferences on $\operatorname{range} f$. Hence, there is a function $\tilde{f}\colon \mathcal{T}^n_{\operatorname{range} f} \to \operatorname{range} f$ such that

$$f(t_1, t_2, \ldots, t_n) = \tilde{f}(p(t_1), p(t_2), \ldots, p(t_n)) = f'(\tau_{\operatorname{range} f}(t_1), \tau_{\operatorname{range} f}(t_2), \ldots, \tau_{\operatorname{range} f}(t_n)). \tag{17}$$

$\tilde{f}$ is surjective and its range hence an interval of $\operatorname{range} f$. Hence, by Weymark 2011, Theorem 5, this function is strategyproof. This implies by (17) that f must also be strategy-proof. □

Proofs for Section 5

Proposition 8 (Linear and quadratic preferences domains are RST)**.** *(a) The quadratic domain is an RST domain.*

(b) The linear preference domain is an RST domain if and only if $(u^x)_{x\in[0,1]}$ are the sub-differentials of a convex function.

Before we come to the proof of Proposition 8, we formalise the way in which we resolve indifferences. Let $\bullet \in \{\text{quad.}, \text{lin.}\}$. Define $\mathcal{T} := [0,1] \times \{0,1\}$ resp. and $\mathcal{K} := [0,1]$.

Define for $(t,s) \in \mathcal{T}$ the preference relation $\preceq_{\bullet}^{(t,s)}$ by

$$k \prec_{\bullet}^{(t,s)} k' \iff u_{\bullet}^{k}(t) < u_{\bullet}^{k'}(t) \\ \vee \left[u_{\bullet}^{k}(t) = u_{\bullet}^{k'}(t) \wedge ((k < k' \wedge s = 1) \vee (k > k' \wedge s = 0))\right].$$

In other words, this is the lexicographic order induced by the quasiorder given by the utility function $u_{\bullet}^{x}(t)$ and the order on $\{0,1\}$ given by $0 < 1$. This clearly yields a preference relation. Furthermore, define the total order $<$ on $\mathcal{T}$ by

$$(x_i, s_i) < (x_i', s_i') \iff x_i < x_i' \vee (x_i = x_i' \wedge s_i < s_i')$$

which is the lexicographic order on the product $[0,1] \times \{0,1\}$.

Proof of Proposition 8. (a) We first consider the quadratic environment. The domain of preferences is regular, as for any $x \in \mathcal{K}$, $\tau_{\mathcal{K}}((x,0)) = x$, compare Figure 6a (on these images, we do not show the indifference-resolving variables $s \in \{0,1\}$). For the single-crossing property, let $k < k' \in \mathcal{K}$ and $(t,s) < (t',s') \in \mathcal{T}$, compare Figure 6c. Let $k \prec^{(t,s)} k'$. There are two cases. We suppress the subscript quad. in the following calculations for ease of notation.

First, let $u^{k}(t) < u^{k'}(t)$. This is equivalent to $-(k-t)^2 < -(k'-t)^2$ which is in turn equivalent to $k'^2 - k^2 < 2y(k'-k)$ by basic algebra. But as $k' - k > 0$ and $t' > t$, one also has $k'^2 - k^2 < 2y'(k'-k)$, which is equivalent to $u^{k}(t') < u^{k'}(t')$. Hence, $k \prec^{t',s'} k'$.

Second, let $u^{k}(t) = u^{k'}(t)$. It the must be that $s = 1$. Then we have $t' > t$. $u^{k}(t) = u^{k'}(t)$ implies by algebra $k'^2 - k^2 = 2t(k'-k) < 2t'(k'-k)$. Re-substituting, $u^{k}(t') < u^{k'}(t')$ and hence $k \prec^{t',s'} k'$.

Third, the domain is top-connected, compare Figure 6e: Let $x_1 < x_2 < x_3 < x_4$ be adjacent in a finite set $A \subseteq \mathcal{K}$. Then one finds that for $t = \frac{x_2+x_3}{2}$, $\tau_A((t,0)) = x_2$ and $\tau_{A\setminus\{x_2\}}((t,0)) = x_3$.[15] The converse holds for $(t,1)$.

(b) We first show that linear preference are single-crossing and tops-connected. For single-crossingness, very similar calculations as in the quadratic case show the result. For tops-connectedness, let $A \subseteq \mathcal{K}$ and $x_1 < x_2 < x_3 < x_4 \in A$ adjacent, compare Figure 6f. In the following, we suppress the subscript lin. Consider type $t :=$

[15]This also holds without the way in which we resolved indifferences: $\tau_A((t-\varepsilon,0)) = x$ and $\tau_{A\setminus\{x\}}((t-\varepsilon,0)) = y$, whenever $\varepsilon < \frac{x_2+x_3}{2} - \frac{x_3+x_1}{2} = \frac{x_2-x_1}{2} > 0$. In Figure 6e $t - \varepsilon > t'$.

$-\frac{b(x_3)-b(x_2)}{a(x_3)-a(x_2)}$, the intersection of u^{x_3} and u^{x_2}. It holds as in the quadratic case that $\tau_A((t,0)) = x_2$ and $\tau_{A\setminus\{x_2\}}((t,0)) = x_3$ The converse holds for $(t,1)$.[16]

We finally show that regularity of the linear preference domain is equivalent to u^x being the sub-differentials of a convex function. For sufficiency, define the function

$$h\colon\ \mathcal{T} \to \mathbb{R}_+, t \mapsto \max_{x\in\mathcal{K}} u^x(t).$$

This is maximum of affine functions, hence convex. By regularity, for any x there must be t such that $h(t) = u^x(t)$. Hence, the functions touch They do not intersect by definition of h. Hence, all the u^x must be sub-differentials of h, compare Boyd and Vandenberghe 2004, (3.8), (6.20).
Conversely, let $u^x(t)$ be the sub-differentials of a convex function. Then by definition of sub-differentials, $\max_{x\in\mathcal{K}} u^x(t) = h(t)$ and $u^x(t) = h(t)$ for some $x \in \mathcal{K}$. Note that by the requirement that a is strictly decreasing and b is strictly decreasing, there are at most two such x. Call them $x_1 < x_2$. But then $(t,0)$ and $(t,1)$ satisfy $\tau_{\mathcal{K}}((t,0)) = x_1$, $\tau_{\mathcal{K}}((t,1)) = x_2$. Hence, the domain is regular.

□

Corollary 10 (Anonymous version of Achuthankutty and Roy 2018, Corollary 6.2)**.** *Assume that $\mathcal{T}$ is an RST domain. Then for any finite $A \subseteq \mathcal{K}$, a social choice function $\mathcal{T}_A^n \to A$ is strategy-proof, surjective and anonymous if and only if it is an generalised median voting rule on $\mathcal{T}_A$.*

Proof of Corollary 10. Achuthankutty and Roy 2018, Corollary 6.2 says that on an RST domain, a strategy-proof, unanimous social choice function is a min-max rule with the additional property that $a_\emptyset = \underline{k}$ and $a_{\{1,2,\ldots,n\}} = \overline{k}$ (we highlight the non-standard terminology—the authors call these just min-max rules). By Proposition 11 and the construction therein, this means that a strategy-proof, unanimous social choice function is an $(n+1)$-parameter generalised median voting rule with the additional property that the largest and smallest phantom ballot lie on $\underline{k}$ resp. $\overline{k}$. By the construction in Lemma 1 such social choice functions are exactly generalised median voting schemes. □

[16] We note that also without our resolution of indifferences, tops-connectedness can be proved. By the monotonicity assumptions on a and b, it is not too hard to show that for any $x_1 < x_2 < x_3 \in \mathcal{K}$, it holds that $\frac{a(x_3)-a(x_1)}{b(x_1)-b(x_3)} < \frac{a(x_3)-a(x_2)}{b(x_2)-b(x_3)}$, that is, the type where u^{x_1} with u^{x_3} intersect is strictly smaller than the intersection type of u^{x_2} with u^{x_3}. In this case, $(\tilde{t},0)$ with $\tilde{t} \in (t',t)$, where t' is the intersection of u^{x_1} with u^{x_3} has the property that $\tau_A((\tilde{t},0)) = x_3$, $\tau_{A\setminus\{x_3\}}((\tilde{t},0)) = x_2$, compare Figure 6f. Similar conditions hold for the converse.

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Additional Proofs

Corollary 3 (Range of first-best mechanisms)**.** $|\text{range}\, g_{\text{first-best}}| \geq (k-1)n+1$ *and this bound is tight.*

Proof of Corollary 3. Let $f_1 < f_2 < \cdots < f_k$ be the sorted range of $f^{\uparrow}$ and let $\mathcal{M} = \{m_i\}_{i=1}^{k}$ such that $f^{\uparrow}(m_i) = f_i$, $i = 1, 2, \ldots, n$. Consider the following elements of $\mathcal{M}^n$:

$$\mathbf{m}_{i,j} = (\underbrace{m_i, m_i, \ldots, m_i}_{j \text{ times}}, \underbrace{m_{i+1}, m_{i+1}, \ldots, m_{i+1}}_{n-j \text{ times}})$$

For all choices (i,j) such that $i = 0, 2, \ldots, n-1$ and $j = 1, 2, \ldots, k-1$ and $(i,j) = (n, k-1)$, all

$$g_{\text{first-best}}(\mathbf{m}_{ij})$$

are distinct. Note that these are $n(k-1)+1$ different indices. To prove this claim, let $(i,j) \neq (i',j')$ be two such tuples of indices. There are two cases.

(a) Case $j \neq j'$. Without loss, let $j > j'$. Then $g_{\text{first-best}}(\mathbf{m}_{i'j'}) \in [f_{j'}, f_{j'+1})$ and $g_{\text{first-best}}(\mathbf{m}_{ij}) \in [f_j, f_{j+1})$ if $j \neq k-1$ and $g_{\text{first-best}}(\mathbf{m}_{ij}) \in [f_j, f_{j+1}]$ if $j = k$. In any case, these are disjoint intervals and $g_{\text{first-best}}(\mathbf{m}_{ij}) \neq g_{\text{first-best}}(\mathbf{m}_{i'j'})$

(b) Case $j = j, i \neq i'$. Without loss, let $i > i'$. Then by definition of $\mathbf{m}_{ij}$

$$g_{\text{first-best}}(\mathbf{m}_{ij}) - g_{\text{first-best}}(\mathbf{m}_{i'j'}) = (j-j')f_{i+1} - (j-j')f_i = (j-j')(f_{i+1} - f_i) > 0.$$

Hence, in particular $g_{\text{first-best}}(\mathbf{m}_{ij}) \neq g_{\text{first-best}}(\mathbf{m}_{i'j'})$.

This bound is e.g. attained for the uniform distribution on the unit interval. The uniform quantiser (f in the proof of Proposition 2) is the unique optimal quantiser in this case (cf. Bergemann, Shen, Xu, and E. M. Yeh 2011, (5) and Example 1). Tedious but straightforward computation shows that in this case $g_{\text{first-best}}(\mathbf{m}_{ij})$ for the above choices of indices exhausts all of the range of $g_{\text{first-best}}$. □

Proposition 6 (Welfare comparison for second-best mechanism)**.** *Assume quadratic preferences and $n = 2$. There is a distribution F such that the following holds:*

Let g be any unanimous, anonymous, strategy-proof mechanism g in a model without communication restriction, i.e. a generalised median voting rule, and let $(g_k)_{k \in \mathcal{N}}$ be any sequence of embedded generalised median voting rules $g_k \colon \mathcal{M}^n \to \mathcal{K}$ with $|\mathcal{M}| = k$. Then

$$W(F,g) - W(F,g_k) \in \Omega(k^{-1}).$$

where Ω is the Landau symbol for at most as fast convergence.

Proof of Proposition 6. We choose F to be the uniform distribution on the unit square $[0,1]^2$. First note that as in (10), we can decompose both g and g_k's linearly into

$$-W(F,g) = \frac{n-1}{n}\sigma^2 + \frac{1}{n}\mathbb{E}[(g(t_1,t_2)-\bar{t})^2]$$
$$-W(F,g_k) = \frac{n-1}{n}\sigma^2 + \frac{1}{n}\mathbb{E}[(g_k(t_1,t_2)-\bar{t})^2]$$

Denote g's phantom ballot by α_1 and g_k's phantom ballot by α_1^k. To prove the asymptotic bound, we must lower bound $W(F,g) - W(F,g_k)$, which is upon inserting definitions

$$\frac{1}{2}\mathbb{E}\left[\left(\text{med}\{\iota(s(t_1)),\iota(s(t_2)),\alpha_1^k\} - \frac{t_1+t_2}{2}\right)^2\right] - \mathbb{E}\left[\left(\text{med}\{t_1,t_2,\alpha_1\} - \frac{t_1+t_2}{2}\right)^2\right]$$
$$= \mathbb{E}\left[(\text{med}\{t_1,t_2,\alpha_1\} - \text{med}\{\iota(s(t_1)),\iota(s(t_2)),\alpha_1\})\frac{t_1+t_2}{2}\right]$$
$$+\frac{1}{2}\mathbb{E}\left[\text{med}\{\iota(s(t_1)),\iota(s(t_2)),\alpha_1\}^2 - \text{med}\{t_1,t_2,\alpha_1\}^2\right]. \quad (18)$$

Note that we can assume $\alpha_1^k, \alpha_1 \leq \frac{1}{2}$ or $\alpha_1^k, \alpha_1 \geq \frac{1}{2}$ as for the uniform distribution, the generalised median rules with phantom ballots α_1 and $1-\alpha_1$ yield the same welfare.

As the integrand in (18) is non-negative, to prove a lower bound, we may restrict $(t_1, t_2$ to a subset of $[0,1]^2$. Restrict the values of the random variables (t_1,t_2) to the square $\frac{1}{2} \leq t_1 \leq \frac{3}{4}$, $\frac{7}{8} \leq t_2 \leq 1$ if $\alpha_1, \alpha_1^k \leq \frac{1}{2}$ resp. $\frac{1}{4} \leq t_1 \leq \frac{1}{2}$, $0 \leq t_2 \leq \frac{1}{8}$ if $\alpha_1, \alpha_1^k \geq \frac{1}{2}$. We treat exemplary the further case. The latter case can be shown similarly. Let g_k have embedding ι and let $x_0 \leq \frac{1}{2} < x_1 \leq x_2 \leq \cdots \leq x_{\ell-1} < \frac{3}{4} \leq x_\ell$ be the larger part of the range of ι. As welfare of g_k only increases if more messages are allowed, i.e. if the range of ι is larger, we can also assume that there is $x_0 = \frac{1}{2}, x_\ell = \frac{3}{4} \in \text{range}\,\iota$. We restrict t_1 further to the set

$$t_1 \in \bigcup_{i=0}^{\ell-1}\left[x_i, \frac{x_i+x_{i+1}}{2}\right), \quad (19)$$

where $x_{\ell+1} = 1$. In other words, we require $t_i > \iota(s(t_1))$, $i = 1,2$. For ease of notation, we do not write the indicator functions of the set we restricted (t_1,t_2) to. (18) simplifies to

$$\mathbb{E}_{t_1,t_2}\left[(t_1 - \iota(s(t_1)))\frac{t_1+t_2}{2}\right] + \frac{1}{2}\mathbb{E}_{t_1,t_2}[\iota(s(t_1))^2 - t_1^2]$$

By basic algebra,

$$\mathbb{E}_{t_1,t_2}\left[(t_1 - \iota(s(t_1)))\frac{t_1+t_2}{2}\right] + \frac{1}{2}\mathbb{E}_{t_1,t_2}[\iota(s(t_1))^2 - t_1^2]$$

$$= 2^{-1}\mathbb{E}_{t_1,t_2}[(t_1 - \iota(s(t_1)))(t_1 + t_2 - (t_1 + \iota(s(t_1)))]$$
$$= 2^{-1}\mathbb{E}_{t_1,t_2}[(t_1 - \iota(s(t_1)))(t_2 - \iota(s(t_1))]$$

As $\frac{1}{2} \leq t_1 \leq \frac{3}{4}$, $\frac{7}{8} \leq t_2 \leq 1$ and (19), $t_2 - \iota(s(t_1)) \geq t_2 - t_1 \geq \frac{1}{8}$, hence

$$\begin{aligned} 2^{-1}\mathbb{E}_{t_1,t_2}[(t_1 - \iota(s(t_1)))(t_2 - \iota(s(t_1))] &\geq 2^{-4}\mathbb{E}_{t_1,t_2}[t_1 - \iota(s(t_1))] \\ &= 2^{-7}\mathbb{E}_{t_1}[t_1 - \iota(s(t_1))], \end{aligned}$$

where the latter equation is integrating out t_2, knowing, it takes values in a set of measure $\frac{1}{8}$. Using the law of total probability, $\mathbb{P}[\iota(s(t_1)) = x_i] = \frac{x_{i+1}+x_i}{2} - x_i = \frac{x_{i+1}-x_i}{2}$ and $\mathbb{E}_{t_1}[t_1 - \iota(s(t_1))|\iota(s(t_1)) = x_i] = \frac{1}{2}(\frac{x_{i+1}+x_i}{2} - x_i) = \frac{x_{i+1}-x_i}{4}$,

$$\begin{aligned} 2^{-7}\mathbb{E}_{t_1}[t_1 - \iota(s(t_1))] &= 2^{-7}\sum_{i=0}^{\ell-1}\mathbb{E}_{t_1}[t_1 - \iota(s(t_1))|\iota(s(t_1)) = x_i]\mathbb{P}[\iota(s(t_1)) = x_i] \\ &= 2^{-7}\sum_{i=0}^{\ell-1}\frac{(x_{i+1} - x_i)^2}{8} \\ &= 2^{-7}\ell\sum_{i=0}^{\ell-1}\frac{1}{\ell}\frac{(x_{i+1} - x_i)^2}{8} \end{aligned}$$

where the latter is a one-multiplication. Now by Jensen's inequality,

$$\begin{aligned} 2^{-10}\ell\sum_{i=0}^{\ell-1}\frac{1}{\ell}(x_{i+1} - x_i)^2 &\geq 2^{-10}\ell\left(\frac{1}{\ell}\sum_{i=0}^{\ell-1}x_{i+1} - x_i\right)^2 \\ &\geq 2^{-10}\frac{\ell}{\ell^2}(x_\ell - x_0)^2 \\ &\geq 2^{-14}\frac{1}{\ell} \geq 2^{-14}\frac{1}{k}. \end{aligned}$$

Here, the second to last step is a telescopic sum. □

Proposition. *Let g be a welfare maximising unanimous, anonymous, strategy-proof mechanism g in a model without communication restriction, i.e. a generalised median voting rule and F any distribution on $[0,1]$. Then there is a sequence $(g_k)_{k\in\mathcal{N}}$ of embedded generalised median voting rules $g_k\colon \mathcal{M}^n \to \mathcal{K}$ with $|\mathcal{M}| = k$ such that*

$$W(F, g) - W(F, g_k) \in o(1).$$

Proof of . Let f be a welfare maximising generalised median voting rule and let the phantom ballots be $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{K}$ and. Let g_k be the embedded generalised median voting rule with range $\iota_k = \{\frac{1}{k+1}, \frac{2}{k+1}, \ldots, \frac{k}{k+1}\}$.

Fix $t_1, t_2, \ldots, t_n \in \mathcal{T}$. Note that $\iota_k(t_i) \to t_i$. Furthermore, the function

$$h_{t_1,t_2,\ldots,t_n} \colon [0,1]^n \to [0,1], (x_1, x_2, \ldots, x_n) \mapsto \sum_{i=1}^{n} (t_i - \text{med}\{x_1, x_2, \ldots, x_n, \alpha_1, \alpha_2, \ldots, \alpha_n\})^2$$

is continuous. In particular,

$$h_{t_1,t_2,\ldots,t_n}(\iota_k(t_1), \iota_k(t_2), \ldots, \iota_k(t_n)) \xrightarrow{k\to\infty} h_{t_1,t_2,\ldots,t_n}(t_1, t_2, \ldots, t_n).$$

Furthermore, $h_{t_1,t_2,\ldots,t_n}(\iota_k(t_1), \iota_k(t_2), \ldots, \iota_k(t_n)) - h_{t_1,t_2,\ldots,t_n}(t_1, t_2, \ldots, t_n)$ is bounded by t. Hence,

$$W(F, f) - W(F, g_k) = \mathbb{E}[h_{t_1,t_2,\ldots,t_n}(\iota_k(t_1), \iota_k(t_2), \ldots, \iota_k(t_n)) - h_{t_1,t_2,\ldots,t_n}(t_1, t_2, \ldots, t_n)] \to 0,$$

by the dominated convergence theorem Billingsley 2008, Theorem 16.4. □

Proposition 11 (Anonymous min-max is generalised median)**.** *Let $\mathcal{T}$ be regular. Then a min-max social choice function is anonymous if and only if it is an $(n+1)$-parameter generalised median voting scheme.*[17]

Proof of Proposition 11. By the definition of the median, an $(n+1)$-parameter generalised median voting rule is anonymous. Conversly, let f be a min-max social choice function. By regularity of the domain, there are $\underline{t}$ and $\bar{t}$ such that $\tau_{\mathcal{K}}(\underline{t}) = \underline{k}$ and $\tau_{\mathcal{K}}(\bar{t}) = \bar{k}$. Then let $\mathbf{t}_S \in \mathcal{T}^n$ such that $t_i = \underline{t}$ for $i \in S$ and $t_i = \bar{t}$ for $i \notin S$. Then one checks that (compare Weymark 2011, Proposition 2)

$$f(\mathbf{t}_S) = a_S.$$

Note that for any S' with $|S|$ there is $\pi \in S_n$ such that $\pi(S) = S'$. Denote by $\mathbf{t}_S^\pi$ the vector that permutes the entries of $\mathbf{t}_S$ by π. Then by anonymity

$$a_S = g(\mathbf{t}_S) = g(\mathbf{t}_S^\pi) = a_{S'}$$

Hence, a_S only depends on S and we can define $b_{|S|} = a_S$. Then

$$g(\mathbf{t}) = \min_{S\subseteq\{1,2,\ldots,n\}} \max_{i\in S}\{\tau_{\mathcal{K}}(t_i), b_{|S|}\} = \text{med}\{\tau_{\mathcal{K}}(t_1), \tau_{\mathcal{K}}(t_2), \ldots, \tau_{\mathcal{K}}(t_n), b_0, b_1, \ldots, b_n\}$$

where the last equality is by Weymark 2011, Proposition 7. Hence, g is a generalised median voting rule. □

Proposition (RST domains: property A)**.** *RST domains satisfy property A.*

[17] For the definition of $(n+1)$-parameter generalised median voting rules see p. 31.

Proof. Surjectivity follows immediately by the assumption of regularity and the observation that regularity of $\mathcal{T}$ implies regularity of $\mathcal{T}_A$ for any $A \subseteq \mathcal{T}$. For monotonicity, let $t < t' \in \mathcal{T}$. Call $k := \tau_{\mathcal{F}}(t)$, $k' := \tau_{\mathcal{F}}(t')$ and assume $k > k'$ for contradiction. By definition of τ_A, $\tau_A(t) \succeq^t \tau_A(t')$ and $\tau_{\mathcal{F}}(t) \preceq^{t'} \tau_{\mathcal{F}}(t')$. Substituting, $k \succeq^t k'$ and $k \preceq^{t'} k'$. By single-crossing also $k \succeq^{t'} k'$. This is a contradiction by antisymmetry of $\preceq^{t'}$. $\square$

Proposition. *Let F be the uniform distribution on $[0,1]$ and $k = 2$. Then the first-best mechanism is symmetric BIC .*

Proof. Clearly, the strategies for agents in the first-best are symmetric. By Bergemann, Shen, Xu, and E. M. Yeh 2011, Example 1 we know that the first-best mechanism will be (up to F-zero sets)

$$s_i(t) = \begin{cases} 0 & t \leq \frac{1}{2} \\ 1 & t > \frac{1}{2} \end{cases} \qquad g(m_1, m_2, \ldots, m_n) = \frac{1}{4n}\sum_{i=1}^{n} m_i + \frac{3}{4n}\left(n - \sum_{i=1}^{n} m_i\right)$$
$$= \frac{1}{4} + \frac{1}{2n}\sum_{i=1}^{n} m_i$$

Denote by $\mathrm{EU}^t(m)$ the expected utility of an agent of type t that sends message t. Then $\mathrm{EU}^t(0) = \mathbb{E}[-(\frac{1}{4}+\frac{1}{2n}X-t)^2]$ and $\mathrm{EU}^t(1) = \mathbb{E}[-(\frac{1}{4}+\frac{1}{2n}+\frac{1}{2n}X-t)^2]$, where $X = \mathrm{Bin}(n-1, \frac{1}{2})$ is binomially distributed with parameters $n-1$ and $\frac{1}{2}$. Then, tedious but straightforward computations shows

$$\begin{aligned}
\mathrm{EU}^t(1) - \mathrm{EU}^t(0) &= \mathbb{E}\left[-\frac{1}{4n^2} - 2\left(\frac{1}{4} + \frac{1}{2n}X - t\right)\frac{1}{2n}\right] \\
&= -\frac{1}{4n^2} - \frac{n}{4n^2} - \frac{2\mathbb{E}[X]}{4n^2} + \frac{4nt}{4n^2} \\
&= \frac{-1 - n - (n-1) + 4nt}{4n^2} \leq 0 \iff t \leq \frac{1}{2}
\end{aligned}$$

which shows that the strategy s_i is expected utility maximising for any $i = 1, 2, \ldots, n$ and hence $(s_1, s_2, \ldots, s_n)$ forms a Bayesian equilibrium. $\square$

For $n > 2$ but still the uniform distribution, the proof is computationally more demanding. We leave this for further work.

Lemma 3. *If $g : \mathcal{M}^n \to \mathcal{K}$ is anonymous, non-wasteful and dominant strategy implementable by strategies $s_1, s_2, \ldots, s_n \colon \mathcal{T} \to \mathcal{M}$, then $s_i = s_{i'}, 1 \leq i, i' \leq n$.*

Proof of Lemma 3. Assume $s_i(t) \neq s_{i'}(t)$ for some $t \in \mathcal{T}$ and $1 \leq i, i' \leq n$. Then, as we assume strict preference relations, it must be that

$$g(s_i(t), m_{-i}) \succeq^t g(s_{i'}(t), m_{-i}) = g(s_{i'}(t), m_{-i'}) \succeq^t g(s_i(t), m_{-i'}) = g(s_i(t), m_{-i}),$$

where the first follows as s_i is a best response for player i, the second by anonymity, the third, as $s_{i'}$ is a best response for player i' and the final equality again by anonymity. Hence, all must be equal. In particular for any m_i, $g(s_i(t), m_{-i}) = g(s_{i'}(t), m_i)$. By non-wastefulness, $s_i(t) = s_{i'}(t)$. $\square$

Lemma 4. *Let $A \subseteq \mathcal{K}$. Let $x, y \in A$ be adjacent, i.e. there is no $z \in A$ such that $x < z < y$. Let $\alpha_1, \ldots, \alpha_{n-1} \in [\underline{k}, x] \cup [y, \overline{k}]$ and $i = 1, 2, \ldots, n$. Then for $h\colon A \to A, (a_1, a_2 \ldots, a_n) \mapsto \operatorname{med}\{a_1, a_2, \ldots, a_n, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ there is $a_{-i} \in \mathcal{A}^{n-1}$ such that*

$$x = h(x, a_{-i}) \neq h(y, a_{-i}) = y.$$

Proof of Lemma 4. Let

$$\underline{\alpha} := |\{i = 1, 2, \ldots, n | \alpha_i \in [\underline{k}, x]\}|$$

and

$$\overline{\alpha} := \left|\{i = 1, 2, \ldots, n | \alpha_i \in [y, \overline{k}]\}\right|.$$

Then $\underline{\alpha} + \overline{\alpha} = n - 1$. Define a_{-i} such that $|\overline{\alpha} - \underline{\alpha}|$ of the entries are equal to x (y) if $\overline{\alpha} > \underline{\alpha}$ $(\overline{\alpha} \leq \underline{\alpha})$ and and of the rest, $\frac{n-1-|\overline{\alpha}-\underline{\alpha}|}{2}$ (note that this is an integer) are equal to x resp. y. Note that the median does not change when replacing all $\alpha_i, i \in \{i = 1, 2, \ldots, n | \alpha_i \in [\underline{k}, x]\}$ with x and all $\alpha_{i'}$, $i' \in \{i = 1, 2, \ldots, n | \alpha_i \in [y, \overline{k}]\}$ with y. Then for the case $\overline{\alpha} \geq \underline{\alpha}$ (the converse direction is similar)

$$h(x, a_{-i}) = \operatorname{med}\{x, \underbrace{x, \ldots, x}_{\underline{\alpha}}, \underbrace{y, \ldots, y}_{\overline{\alpha}}, \underbrace{x, \ldots, x}_{\overline{\alpha}-\underline{\alpha}}, \underbrace{x, \ldots, x}_{\frac{n+1-|\overline{\alpha}-\underline{\alpha}|}{2}}, \underbrace{y, \ldots, y}_{\frac{n+1-|\overline{\alpha}\ -\underline{\alpha}|}{2}}\} = x$$

$$h(y, a_{-i}) = \operatorname{med}\{y, \underbrace{x, \ldots, x}_{\underline{\alpha}}, \underbrace{y, \ldots, y}_{\overline{\alpha}}, \underbrace{x, \ldots, x}_{\overline{\alpha}-\underline{\alpha}}, \underbrace{x, \ldots, x}_{\frac{n+1-|\overline{\alpha}-\underline{\alpha}|}{2}}, \underbrace{y, \ldots, y}_{\frac{n+1-|\overline{\alpha}\ -\underline{\alpha}|}{2}}\} = y.$$

$\square$

Statement of Authorship

I hereby confirm that the work presented has been performed and interpreted solely by myself except for where I explicitly identified the contrary. I assure that this work has not been presented in any other form for the fulfilment of any other degree or qualification. Ideas taken from other works in letter and in spirit are identified in every single case.

Bonn, August 31, 2018

Andreas Alexander HAUPT